HW #10 (221A), due Nov 19, 4pm

1. We would like to find the ground-state wave function of a particle in the potential $V = 50(e^{-x}-1)^2$ with $m = 1$, $\hbar = 1$. In this case, The true ground-state energy is known to be $E_0 = 39/8$. Plot the form of the potential. Find a variational wave function that comes within 5% of the true energy.

2. Neutrinos oscillate via the Hamiltonian,

$$H = \sqrt{c^2\vec{p}^2 + m^2c^4} \simeq c|\vec{p}| + \frac{m^2c^3}{2|\vec{p}|}, \tag{1}$$

where the m^2 is a three-by-three matrix

$$m^2 = UDU^\dagger, \qquad D = \mathrm{diag}(m_1^2, m_2^2, m_3^2). \tag{2}$$

We parameterize the unitarity matrix U with four parameters θ_{12}, θ_{13}, θ_{23}, and δ as

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3}$$

where other five unimportant phases are already dropped. We ignore the spin degrees of freedom. The notation is $s_{12} = \sin\theta_{12}$, $c_{23} = \cos\theta_{23}$, etc. Three species of neutrinos are represented by

$$|\nu_e(\vec{p})\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes |\vec{p}\rangle, \quad |\nu_\mu(\vec{p})\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes |\vec{p}\rangle, \quad |\nu_\tau(\vec{p})\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes |\vec{p}\rangle. \tag{4}$$

(a) First consider the two-by-two case, by taking the limit $\theta_{12} = \theta_{13} = 0$. Show that the survival probability is

$$P(\nu_\mu \to \nu_\mu, t) = |\langle \nu_\mu | e^{-iHt/\hbar} | \nu_\mu \rangle|^2 = 1 - \sin^2 2\theta_{23} \sin^2 \frac{(m_3^2 - m_2^2)c^3 t}{4\hbar|\vec{p}|}. \tag{5}$$

(b) For $|\vec{p}| = 1$ GeV/c, and $m_3^2 - m_2^2 = 2.5 \times 10^{-3}$ eV$^2/c^4$, plot the survival probability as a function of the flight distance $L = ct$ so that the oscillatory behavior can be seen.

(c) Consider full three states, and show that $P(\nu_\mu \to \nu_e) \neq P(\nu_e \to \nu_\mu)$ if $\delta \neq 0$.

(d) Show that the time-reversal invariance would predict $P(\nu_\mu \to \nu_e) = P(\nu_e \to \nu_\mu)$, and hence $\delta \neq 0$ violates the time reversal invariance.