HW #3

1. Spin Matrices

We use the spin operators represented in the bases where S_z is diagonal:

$$S_x = \frac{\hbar}{2} \{\{0, 1\}, \{1, 0\}\}; S_y = \frac{\hbar}{2} \{\{0, -I\}, \{I, 0\}\}; S_z = \frac{\hbar}{2} \{\{1, 0\}, \{0, -1\}\};$$

S_x // MatrixForm

$$\begin{pmatrix} 0 & \frac{\hbar}{2} \\ \frac{\hbar}{2} & 0 \end{pmatrix}$$

S_y // MatrixForm

$$\begin{pmatrix} 0 & -\frac{i\hbar}{2} \\ \frac{i\hbar}{2} & 0 \end{pmatrix}$$

S_z // MatrixForm

$$\begin{pmatrix} \frac{\hbar}{2} & 0 \\ 0 & -\frac{\hbar}{2} \end{pmatrix}$$

(a) Obviously two matrices commute when they are the same: $i = j$. Also, it is obvious that $[S_i, S_j]$ is anti-symmetric in $i \longleftrightarrow j$ because $[S_j, S_i] = -[S_i, S_j]$. Therefore, it only reamains to verify

$S_x . S_y - S_y . S_x - I \hbar S_z$

{{0, 0}, {0, 0}}

$S_y . S_z - S_z . S_y - I \hbar S_x$

{{0, 0}, {0, 0}}

$S_z . S_x - S_x . S_z - I \hbar S_y$

{{0, 0}, {0, 0}}

(b) We define $\vec{n} = (\sin\theta \cos\phi,\ \sin\theta \sin\phi,\ \cos\theta)$

n_x = Sin[θ] Cos[ϕ]; n_y = Sin[θ] Sin[ϕ]; n_z = Cos[θ]

Cos[θ]

S_n = Simplify[n_x S_x + n_y S_y + n_z S_z]

$$\left\{\left\{\frac{1}{2}\hbar \mathrm{Cos}[\theta], \frac{1}{2}\hbar \mathrm{Sin}[\theta] (\mathrm{Cos}[\phi] - i \mathrm{Sin}[\phi])\right\}, \left\{\frac{1}{2}\hbar \mathrm{Sin}[\theta] (\mathrm{Cos}[\phi] + i \mathrm{Sin}[\phi]), -\frac{1}{2}\hbar \mathrm{Cos}[\theta]\right\}\right\}$$

```
Eigensystem[S_n]
```

$$\left\{\left\{-\frac{\sqrt{\hbar^2}}{2}, \frac{\sqrt{\hbar^2}}{2}\right\}, \left\{\left\{\frac{\left(-\sqrt{\hbar^2}+\hbar\,\mathrm{Cos}[\theta]\right)\mathrm{Csc}[\theta]}{\hbar\,(\mathrm{Cos}[\phi]+\mathrm{i}\,\mathrm{Sin}[\phi])}, 1\right\}, \left\{\frac{\left(\sqrt{\hbar^2}+\hbar\,\mathrm{Cos}[\theta]\right)\mathrm{Csc}[\theta]}{\hbar\,(\mathrm{Cos}[\phi]+\mathrm{i}\,\mathrm{Sin}[\phi])}, 1\right\}\right\}\right\}$$

```
PowerExpand[%]
```

$$\left\{\left\{-\frac{\hbar}{2}, \frac{\hbar}{2}\right\}, \left\{\left\{\frac{(-\hbar+\hbar\,\mathrm{Cos}[\theta])\,\mathrm{Csc}[\theta]}{\hbar\,(\mathrm{Cos}[\phi]+\mathrm{i}\,\mathrm{Sin}[\phi])}, 1\right\}, \left\{\frac{(\hbar+\hbar\,\mathrm{Cos}[\theta])\,\mathrm{Csc}[\theta]}{\hbar\,(\mathrm{Cos}[\phi]+\mathrm{i}\,\mathrm{Sin}[\phi])}, 1\right\}\right\}\right\}$$

```
Simplify[%]
```

$$\left\{\left\{-\frac{\hbar}{2}, \frac{\hbar}{2}\right\}, \left\{\left\{(-\mathrm{Cos}[\phi]+\mathrm{i}\,\mathrm{Sin}[\phi])\,\mathrm{Tan}\left[\frac{\theta}{2}\right], 1\right\}, \left\{\mathrm{Cot}\left[\frac{\theta}{2}\right](\mathrm{Cos}[\phi]-\mathrm{i}\,\mathrm{Sin}[\phi]), 1\right\}\right\}\right\}$$

Therefore, one can take the the normalized eigenstates to be $\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\,e^{i\phi}\end{pmatrix}$ with eigenvalue $+\frac{\hbar}{2}$ and $\begin{pmatrix}-\sin\frac{\theta}{2}\,e^{-i\phi}\\ \cos\frac{\theta}{2}\end{pmatrix}$ with eigenvalue $-\frac{\hbar}{2}$. The state with spin along the $\vec{n}$ direction is the former, and its probability to have the positive S_z when measured is simply given by $|\langle S_z=+\frac{\hbar}{2}\,|\,S_n=+\frac{\hbar}{2}\rangle|^2 = \left|(1,0)\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\,e^{i\phi}\end{pmatrix}\right|^2 = \cos^2\frac{\theta}{2}$.

(c) Between $\vec{n} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)$ and $\vec{n}' = (\sin\theta'\cos\phi',\ \sin\theta'\sin\phi',\ \cos\theta')$, the probability is $|\langle S_{n'}=+\frac{\hbar}{2}\,|\,S_n=+\frac{\hbar}{2}\rangle|^2 = \left|(\cos\frac{\theta'}{2}, \sin\frac{\theta'}{2}\,e^{-i\phi'})\begin{pmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\,e^{i\phi}\end{pmatrix}\right|^2 = |\cos\frac{\theta'}{2}\cos\frac{\theta}{2}+\sin\frac{\theta'}{2}\,e^{-i\phi'}\sin\frac{\theta}{2}\,e^{i\phi}|^2 =$

$$\cos^2\frac{\theta'}{2}\cos^2\frac{\theta}{2}+\sin^2\frac{\theta'}{2}\sin^2\frac{\theta}{2}+2\cos\frac{\theta'}{2}\cos\frac{\theta}{2}\sin\frac{\theta'}{2}\sin\frac{\theta}{2}\cos(\phi-\phi')$$

```
TrigExpand[
 Cos[θ1/2]^2 Cos[θ2/2]^2 + Sin[θ1/2]^2 Sin[θ2/2]^2 + 2 Cos[θ1/2] Cos[θ2/2] Sin[θ1/2] Sin[θ2/2] Cos[ϕ1 - ϕ2]]
```

$$\frac{1}{2}+\frac{1}{2}\mathrm{Cos}\left[\frac{\theta_1}{2}\right]^2\mathrm{Cos}\left[\frac{\theta_2}{2}\right]^2-\frac{1}{2}\mathrm{Cos}\left[\frac{\theta_2}{2}\right]^2\mathrm{Sin}\left[\frac{\theta_1}{2}\right]^2+2\,\mathrm{Cos}\left[\frac{\theta_1}{2}\right]\mathrm{Cos}\left[\frac{\theta_2}{2}\right]\mathrm{Cos}[\phi_1]\,\mathrm{Cos}[\phi_2]\,\mathrm{Sin}\left[\frac{\theta_1}{2}\right]\mathrm{Sin}\left[\frac{\theta_2}{2}\right]-\frac{1}{2}\mathrm{Cos}\left[\frac{\theta_1}{2}\right]^2\mathrm{Sin}\left[\frac{\theta_2}{2}\right]^2+\frac{1}{2}\mathrm{Sin}\left[\frac{\theta_1}{2}\right]^2\mathrm{Sin}\left[\frac{\theta_2}{2}\right]^2+2\,\mathrm{Cos}\left[\frac{\theta_1}{2}\right]\mathrm{Cos}\left[\frac{\theta_2}{2}\right]\mathrm{Sin}\left[\frac{\theta_1}{2}\right]\mathrm{Sin}\left[\frac{\theta_2}{2}\right]\mathrm{Sin}[\phi_1]\,\mathrm{Sin}[\phi_2]$$

```
Simplify[%]
```

$$\frac{1}{2}(1+\mathrm{Cos}[\theta_1]\,\mathrm{Cos}[\theta_2]+\mathrm{Cos}[\phi_1]\,\mathrm{Cos}[\phi_2]\,\mathrm{Sin}[\theta_1]\,\mathrm{Sin}[\theta_2]+\mathrm{Sin}[\theta_1]\,\mathrm{Sin}[\theta_2]\,\mathrm{Sin}[\phi_1]\,\mathrm{Sin}[\phi_2])$$

This is nothing but $\frac{1}{2}\left(1+\vec{n}\cdot\vec{n}'\right) = \frac{1}{2}(1+\cos\eta) = \cos^2\frac{\eta}{2}$, where η is the angle between two vectors, as expected from the rotational invariance.

2. Sloppy Hydrogen Atom

According to the problem,

$$\text{Energy} = \frac{1}{2\,m}\left(\frac{\hbar}{d}\right)^2 - \frac{Z\,e^2}{d}$$

$$-\frac{e^2\,Z}{d} + \frac{\hbar^2}{2\,d^2\,m}$$

```
Solve[D[Energy, d] == 0, d]
```

$$\left\{\left\{d \to \frac{\hbar^2}{e^2\,m\,Z}\right\}\right\}$$

```
Simplify[Energy /. %[[1]]]
```

$$-\frac{e^4\,m\,Z^2}{2\,\hbar^2}$$

This actually agrees with the exact result. (One should be cautioned, however, that the agreement with the exact result is a coincidence for this particular example.)

3. Classical Uncertainty Principle

(a) The Maxwell's equations in vacuum are given by

$$\vec{\nabla}\cdot\vec{E} = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

$$\vec{\nabla}\times\vec{E} + \partial_t\vec{B} = 0$$

$$c^2\,\vec{\nabla}\times\vec{B} - \partial_t\vec{E} = 0$$

In this problem, there are only x and t dependence, and the only non-vanishing components are E_y and B_z. Then the Maxwell's equations reduce to

$$\nabla_x E_y + \partial_t B_z = 0$$

$$-c^2\,\nabla_x B_z - \partial_t E_y = 0$$

Putting them together, they reduce to a simple one-dimensional equation,

$$c^2\,\nabla_x{}^2 E_y - \partial_t{}^2 E_y = 0.$$

Any function of the combination $c\,t - x$ satisfies this equation, namely

$$(c^2\,\nabla_x{}^2 - \partial_t{}^2)\,f(c\,t - x) = 0.$$

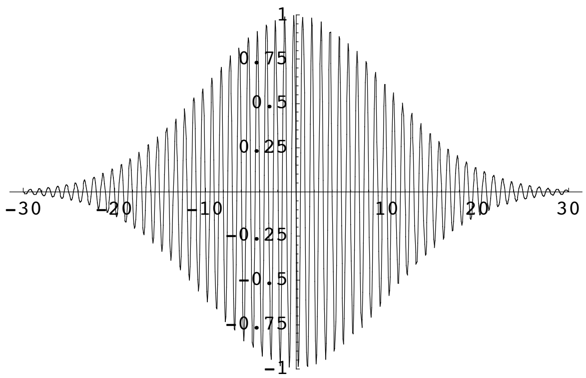
Because the form of E_y given in the problem is a function of $c\,t - x$ only, it solves the Maxwell's equations automatically. The form can be sketched as

To simplify the problem, define $\gamma = 4\,\pi^2\,\nu^2\,\sigma^2\,/\,c^2$. Then the electric field is:

```
Ey[x_, t_] = E0 * Sin[c Sqrt[γ]/σ t - Sqrt[γ]/σ x] E^-(x-c t)^2/(2 σ^2)
```

$$e^{-\frac{(-c\,t+x)^2}{2\,\sigma^2}}\,E0\,\text{Sin}\left[\frac{c\,t\,\sqrt{\gamma}}{\sigma} - \frac{x\,\sqrt{\gamma}}{\sigma}\right]$$

```
Plot[Ey[x, t] /. {t → 0, γ → 400 π^2, E0 → 1, σ → 10}, {x, -30, 30}, PlotRange → {-1, 1}]
```




```
- Graphics -
```

It oscillates just like the plane waves, but is localized. The "uncertainty" is defined using the formula analogous to the quantum mechanical wave function. First the "norm,"

```
norm = Integrate[Ey[x, 0]^2, {x, -∞, ∞}, Assumptions → {γ > 0, σ > 0}]
```

```
General::spell1 : Possible spelling error: new symbol name "norm" is similar to existing symbol "Norm". More…
```

$$\frac{1}{2} e^{-\gamma} (-1 + e^{\gamma}) \, E0^2 \sqrt{\pi} \, \sigma$$

We could set the overall normalization $E_0 = 1$ throughout since it will drop out after taking the norm correctly into account. But we can also leave it in as a check that everything is working correctly.

Next the expectation value

```
Integrate[x * Ey[x, 0]^2, {x, -∞, ∞}, Assumptions → {γ > 0, σ > 0}]
```

$$0$$

OK, this vanishes. Finally the variance,

```
temp = Integrate[x^2 * Ey[x, 0]^2, {x, -∞, ∞}, Assumptions → {γ > 0, σ > 0}]
```

$$\frac{1}{4} e^{-\gamma} E0^2 \sqrt{\pi} \, (-1 + e^{\gamma} + 2\gamma) \, \sigma^3$$

```
variance = temp / norm
```

```
General::spell1 :
 Possible spelling error: new symbol name "variance" is similar to existing symbol "Variance". More…
```

$$\frac{(-1 + e^{\gamma} + 2\gamma) \, \sigma^2}{2 \, (-1 + e^{\gamma})}$$

Note that the E0 dependence did in fact drop out.

```
FullSimplify[variance]
```

$$\frac{1}{2} \left(1 + \frac{2\gamma}{-1 + e^{\gamma}}\right) \sigma^2$$

One can write it as $(\Delta x)^2 = \frac{1}{2}\sigma^2\left(1+\frac{2\gamma}{e^\gamma-1}\right)$, where $\gamma = 4\pi^2\nu^2\sigma^2/c^2$. It is especially simple when $\gamma \gg 1$, when $(\Delta x)^2 = \frac{1}{2}\sigma^2$.

(b) The Fourier transform to the frequency domain as a function of the variable "f" is calculated below. We might as well pick a specific position like "x=0" to evaluate the Fourier transform.

fttemp = Integrate[Ey[0, t] * $E^{I\,2\,\pi\,f\,t}$, {t, -∞, ∞}, Assumptions → {γ > 0, σ > 0, c > 0}]

```
Integrate::gener : Unable to check convergence. More…
```

$$\text{E0 If}\left[f\in\text{Reals},\ \frac{i\,e^{-\frac{(c\sqrt{\gamma}+2f\pi\sigma)^2}{2c^2}}\left(-1+e^{\frac{4f\pi\sqrt{\gamma}\,\sigma}{c}}\right)\sqrt{\frac{\pi}{2}}\,\sigma}{c},\ \text{Integrate}\left[e^{2\,i\,f\,\pi\,t-\frac{c^2t^2}{2\sigma^2}}\,\text{Sin}\left[\frac{c\,t\sqrt{\gamma}}{\sigma}\right],\ \{t,-\infty,\infty\},\ \text{Assumptions}\to c>0\,\&\&\,\gamma>0\,\&\&\,\sigma>0\,\&\&\,f\notin\text{Reals}\right]\right]$$

ft[f_] = fttemp[[1]] * fttemp[[2, 2]]

$$\frac{i\,e^{-\frac{(c\sqrt{\gamma}+2f\pi\sigma)^2}{2c^2}}\left(-1+e^{\frac{4f\pi\sqrt{\gamma}\,\sigma}{c}}\right)\text{E0}\sqrt{\frac{\pi}{2}}\,\sigma}{c}$$

Again starting with the norm (noting that $\text{ft}[f]^*\,\text{ft}[f] = -\text{ft}[f]^\wedge 2$):

norm2 = Integrate[-ft[f]^2, {f, 0, ∞}, Assumptions → {γ > 0, σ > 0, c > 0}]

$$\frac{e^{-\gamma}\,(-1+e^\gamma)\,\text{E0}^2\sqrt{\pi}\,\sigma}{4\,c}$$

Integrate[-ft[f]^2, {f, -∞, ∞}, Assumptions → {γ > 0, σ > 0, c > 0}]

$$\frac{e^{-\gamma}\,(-1+e^\gamma)\,\text{E0}^2\sqrt{\pi}\,\sigma}{2\,c}$$

Next, the average frequency,

exptf = Integrate[f * -ft[f]^2, {f, 0, ∞}, Assumptions → {γ > 0, σ > 0, c > 0}] / norm2

$$\frac{c\,e^\gamma\sqrt{\gamma}\,\text{Erf}[\sqrt{\gamma}]}{2\,(-1+e^\gamma)\,\pi\,\sigma}$$

exptf /. σ → c $\sqrt{\gamma}$ / (2 π ν)

$$\frac{e^\gamma\,\nu\,\text{Erf}[\sqrt{\gamma}]}{-1+e^\gamma}$$

Limit[Erf[$\sqrt{\gamma}$], γ → ∞]

1

One can write it as $\nu\,\frac{\text{Erf}[\gamma^{1/2}]}{1-E^{-\gamma}}$, where $\gamma = 4\pi^2\nu^2\sigma^2/c^2$. It is especially simple when $\gamma \gg 1$, when it reduces to nothing but ν. Finally the variance in the frequency is (remember to normalize!)

```
exptf2 = Integrate[f^2 * -ft[f]^2, {f, 0, ∞}, Assumptions → {γ > 0, σ > 0, c > 0}] / norm2
```

$$\frac{c^2\,(-1+e^{\gamma}\,(1+2\,\gamma))}{8\,(-1+e^{\gamma})\,\pi^2\,\sigma^2}$$

```
FullSimplify[%]
```

$$\frac{c^2\,(-1+e^{\gamma}\,(1+2\,\gamma))}{8\,(-1+e^{\gamma})\,\pi^2\,\sigma^2}$$

```
% /. σ → c √γ / (2 π ν) // Simplify
```

$$\frac{(-1+e^{\gamma}\,(1+2\,\gamma))\,\nu^2}{2\,(-1+e^{\gamma})\,\gamma}$$

```
Limit[%, γ → ∞]
```

$$\nu^2$$

Then the square of the dispersion in the frequency is:

```
dispersion2 = exptf2 - exptf^2 // Simplify
```

$$-\frac{c^2\left(-(-1+e^{\gamma})\,(-1+e^{\gamma}\,(1+2\,\gamma))+2\,e^{2\,\gamma}\,\gamma\,\mathrm{Erf}[\sqrt{\gamma}\,]^2\right)}{8\,(-1+e^{\gamma})^2\,\pi^2\,\sigma^2}$$

```
dispersion2 /. σ → c √γ / (2 π ν) // Simplify
```

$$-\frac{\nu^2\left(-(-1+e^{\gamma})\,(-1+e^{\gamma}\,(1+2\,\gamma))+2\,e^{2\,\gamma}\,\gamma\,\mathrm{Erf}[\sqrt{\gamma}\,]^2\right)}{2\,(-1+e^{\gamma})^2\,\gamma}$$

```
Limit[dispersion2, γ → ∞]
% /. σ → c √γ / (2 π ν)
```

$$\frac{c^2}{8\,\pi^2\,\sigma^2}$$

$$\frac{\nu^2}{2\,\gamma}$$

Namely, $(\Delta f)^2 = \frac{\nu^2\left((1-e^{-\gamma})\,((1-e^{-\gamma})+2\,\gamma)-2\,\gamma\,\mathrm{Erf}[\gamma^{1/2}]^2\right)}{2\,\gamma\,(1-e^{-\gamma})^2}$ which simplifies to $(\Delta f)^2 = \nu^2\,\frac{1}{2\,\gamma} = \frac{c^2}{8\,\pi^2\,\sigma^2}$ when $\gamma \gg 1$. Therefore, $(\Delta x)^2\,(\Delta f)^2 = \frac{c^2}{16\,\pi^2}$.

Once interpreted as a photon, $(\Delta f)^2 = c^2 (\Delta p)^2 / h^2$, and hence $(\Delta x)^2\,(\Delta p)^2 = \frac{\hbar^2}{4}$, as expected.