HW #6

1. Free Particle Propagator

(a)

To simplify the notation, we write $t = t'' - t'$, $x = x'' - x'$.

First of all, because the base kets evolve according to the "wrong sign" Schrödinger equation (see pp. 87-89), $| x', t' \rangle = e^{+i H t'/\hbar} | x', 0\rangle$, $\langle x'', t'' | = \langle x'', 0 | e^{-i H t''/\hbar}$.

Therefore,
$\langle x'', t'' | x', t' \rangle = \langle x'' | e^{-i H (t''-t')/\hbar} | x' \rangle = \int d p \langle x'' | p \rangle \langle p | e^{-i H t/\hbar} | x' \rangle$
$= \int d p \frac{e^{i p x''/\hbar}}{(2 \pi \hbar)^{1/2}} \langle p | e^{-i (p^2/2 m) t/\hbar} | x' \rangle$
$= \int d p \frac{e^{i p x''/\hbar}}{(2 \pi \hbar)^{1/2}} e^{-i (p^2/2 m) t/\hbar} \frac{e^{-i p x'/\hbar}}{(2 \pi \hbar)^{1/2}}$
$= \int d p \frac{e^{i p x/\hbar}}{2 \pi \hbar} e^{-i p^2 t/2 m \hbar}$
$= \int d p \frac{1}{2 \pi \hbar} \exp\left(-i \frac{t}{2 m \hbar} (p - \frac{m x}{t})^2 + i \frac{m x^2}{2 \hbar t}\right)$
$= \frac{1}{2 \pi \hbar} \sqrt{\frac{2 \pi m \hbar}{i t}} e^{i m x^2/2 \hbar t}$
$= \sqrt{\frac{m}{2 \pi i \hbar t}} e^{i m x^2/2 \hbar t}$.

(b)

For the classical trajectory, the velocity is simply $v = \frac{x}{t}$, and hence the action is $S_c = \int \frac{1}{2} m \dot{x}^2 d t = \frac{1}{2} m (\frac{x}{t})^2 t = \frac{m x^2}{2 t}$, and hence the exponent of the propagator is indeed $(i S_c / \hbar)$.

Geometric Optics

(a)

Following the same steps we did for the Schrödinger equation, we first write the Maxwell's equation with $A^0 = e^{i S/\hbar}$:
$\left(\frac{n^2}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right) e^{i S/\hbar} = \left(\frac{n^2}{c^2} \left(\frac{i \ddot{S}}{\hbar} + \left(\frac{i \dot{S}}{\hbar}\right)^2\right) - \frac{i \vec{\nabla}^2 S}{\hbar} - \left(\frac{i \vec{\nabla} S}{\hbar}\right)^2\right) e^{i S/\hbar} = 0.$
In the limit $S \gg \hbar$, we can drop terms of $O(S/\hbar)$ and keep those of $O(S/\hbar)^2$:

$$\frac{n^2}{c^2} \dot{S}^2 - \left(\vec{\nabla} S\right)^2 = 0.$$

This is the "Hamilton-Jacobi" equation.

(Of course, in the Maxwell equation, there is no notion of $\hbar$. What we are doing is a valid approximation when the variation of the phase is very fast compared to the variation of the index of refraction. It is called "eikonal approximation" in optics.)

(b)

Assuming $n(\vec{x}) = n(x)$, the "Hamilton-Jacobi" equation does not have an explicit dependence on y or t. Writing

$$S(x, y, t) = \tilde{S}(x, p_y, E) + p_y\, y - E\, t,$$

the equation becomes

$$\frac{n^2}{c^2} E^2 - \left(\frac{d}{dx}\tilde{S}\right)^2 - p_y{}^2 = 0.$$

Therefore,

$$\frac{d\tilde{S}}{dx} = \sqrt{\frac{n^2}{c^2} E^2 - p_y{}^2}$$

and hence

$$\tilde{S} = \int^x \sqrt{\frac{n(x')^2}{c^2} E^2 - p_y{}^2}\; dx'.$$

Here, the possible x-dependence of the index of refraction is emphasized.

(c)

Using $t = \frac{\partial \tilde{S}}{\partial E}$, $y = -\frac{\partial \tilde{S}}{\partial p_y}$, we find

$$t = \int^x \frac{\frac{n(x')^2}{c^2} E}{\sqrt{\frac{n(x')^2}{c^2} E^2 - p_y{}^2}}\; dx',$$

$$y = \int^x \frac{p_y}{\sqrt{\frac{n(x')^2}{c^2} E^2 - p_y{}^2}}\; dx'.$$

(d)

Assuming that $n(x) = n_1$ for $x < 0$ and $n(x) = n_2$ for $x > 0$, and choosing the lower end of the integration at $x = 0$,

$$y_<(x) = \frac{p_y}{\sqrt{\frac{n_1{}^2}{c^2} E^2 - p_y{}^2}}\, x \text{ for } x < 0$$

and

$$y_>(x) = \frac{p_y}{\sqrt{\frac{n_2{}^2}{c^2} E^2 - p_y{}^2}}\, x \text{ for } x > 0.$$

Using the trigonometric relation $\sin\alpha = \frac{1}{\sqrt{\cot^2\alpha + 1}}$, we find

$$\sin\alpha_< = \frac{1}{\sqrt{\frac{\frac{n_1{}^2}{c^2} E^2 - p_y{}^2}{p_y{}^2} + 1}} = \frac{c\, p_y}{n_1\, E},$$

$$\sin\alpha_> = \frac{1}{\sqrt{\frac{\frac{n_2{}^2}{c^2} E^2 - p_y{}^2}{p_y{}^2} + 1}} = \frac{c\, p_y}{n_2\, E},$$

and hence

$$\frac{\sin\alpha_<}{\sin\alpha_>} = \frac{n_2}{n_1},$$

which is nothing but the Snell's law of refraction.

Bohr-Sommerfeld

We use Eq. (35) of the lecture notes
$\int_{x_{\min}}^{x_{\max}} \sqrt{2\,m(E-V(x))}\;d\,x = (n+\frac{1}{2})\,\pi\,\hbar$,
where $x_{\min}$ and $x_{\max}$ are the classical turning points. For a harmonic oscillator, $V(x) = \frac{1}{2}\,m\,\omega^2\,x^2$, and hence $-x_{\min} = x_{\max} = \sqrt{\frac{2\,E}{m\,\omega^2}}$. The integral is

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Integrate[m ω Sqrt[X² - x²], {x, -X, X}]
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$$\frac{m\,\pi\,X^2\,\omega\,\text{Sign}[X]}{2\,\sqrt{\text{Sign}[X]^2}}$$

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PowerExpand[%]
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$$\frac{1}{2}\,m\,\pi\,X^2\,\omega$$

Here, we used $X = x_{\max}$. Therefore,
$\frac{1}{2}\,m\,\pi\,\frac{2\,E}{m\,\omega^2}\,\omega = (n+\frac{1}{2})\,\pi\,\hbar$,
and hence
$E = (n+\frac{1}{2})\,\hbar\,\omega$.
This result agrees with the exact result, which is an accident for this particular case.