HW #6 (221A), due Oct 8, 4pm

1. Consider a free particle with Hamiltonian $H = p^2/2m$. (a) Show that (Sakurai's Eq. (2.5.16))

$$\langle x'', t''|x', t'\rangle = \sqrt{\frac{m}{2\pi i\hbar(t''-t')}} \exp\left[\frac{im(x''-x')^2}{2\hbar(t''-t')}\right]. \tag{1}$$

   (b) Show that the exponent is nothing but the classical action for the particle to go from x' at t' to x'' at t''.

2. The Maxwell equation (in Lorentz gauge) is

$$\left(\frac{n^2}{c^2}\frac{d^2}{dt^2} - \vec{\nabla}^2\right) A^0(\vec{x}, t) = 0. \tag{2}$$

   Consider it as a "Schrödinger equation" for the "light particle." Here, $n(\vec{x})$ is the index of refraction. Answer the following questions.

   (a) Writing $A^0 = e^{iS/\hbar}$, use "WKB Approximation" to write down the "Hamilton–Jacobi" equation for $S(\vec{x}, t)$.

   (b) Assume that the index of refraction depends only on x. Then we can separate variables t and y (forget z in this problem). Solve for $\tilde{S}(x, p_y, E) = S(x, y, t) - p_y y + Et$ as a function of x in an integral expression in the way you normally do for Hamilton–Jacobi equation.

   (c) Write down integral expressions for t and y.

   (d) Specialize to the case where $n(x) = n_1$ for $x < 0$ and $n(x) = n_2$ for $x > 0$. Show that the trajectory of the "light particle" follows the usual rule of refraction.

   Note The geometrical optics is none other than the "WKB approximation" for the Maxwell's equation.

3. Apply the Bohr–Sommerfeld quantization condition to the harmonic oscillator and work out the energy levels. (See my lecture notes `http://hitoshi.berkeley.edu/221A/WKB.pdf`).