HW #9

1. Rectangular Double-Well Potential

Because the potential is infinite for $|x| > a+b$, the wave function should vanish at $x = \pm(a+b)$. For $a < x < a+b$ and $-a-b < x < -a$, the potential vanishes and the wave function is given simply by a plane wave $e^{\pm i k x}$ with energy $E = \hbar^2 k^2 / 2m$. For $-a < x < a$, the potential is large V_0, and the wave function damps exponentially, $e^{\pm \kappa x}$ with $\hbar^2 \kappa^2 = 2m(V_0 - E) = 2m V_0 - \hbar^2 k^2$. Because the potential is parity-invariant $V(-x) = V(x)$, we expect the ground state to be a symmetric (parity-even) function, with a close excited state with an anti-symmetric (parity-odd) wave function.

For the parity-even ground state, the wave function for $-a < x < a$ must be $\psi(x) = A \cosh \kappa x$. For $a < |x| < a+b$, the wave function must be $\psi(x) = B \sin k(x-a-b)$ to ensure the boundary condition at $|x| = a+b$. For $-a-b < x < -a$, $\psi(x) = \psi(-x) = -B \sin k(x+a+b)$. To match the logarithmic derivative $\psi'(a)/\psi(a)$, we need $\frac{\psi'(a)}{\psi(a)} = \kappa \tanh \kappa a = k \cot k(-b)$. The conditions for $x < 0$ are precisely the same thanks to the parity.

For the parity-odd excited state, the wave function for $-a < x < a$ must be $\psi(x) = A \sinh \kappa x$. For $a < x < a+b$, the wave function must be $\psi(x) = B \sin k(x-a-b)$ to ensure the boundary condition at $|x| = a+b$. For $-a-b < x < -a$, $\psi(x) = -\psi(-x) = B \sin k(x+a+b)$. To match the logarithmic derivative $\psi'(a)/\psi(a)$, we need $\frac{\psi'(a)}{\psi(a)} = \kappa \coth \kappa a = k \cot k(-b)$. The conditions at $x = -a$ are precisely the same thanks to the parity.

First we take the limit of the inifinite potenital barrier $V_0 \to \infty$, and hence $\kappa \to \infty$. Then $\coth \kappa a = \tanh \kappa a = 1$, and the condition is $-k a \cot k b = \kappa a \to \infty$. The only way to satisfy this equation is by taking $k b = (2n+1)\pi$ so that $\cot k b \to -\infty$. The lowest energy is obtained by $k b = \pi$. Namely, to the leading order in large V_0, $k = \frac{\pi}{b}$ and $E = \frac{\hbar^2 \pi^2}{2 m b^2}$ for both parity-even and odd states. This makes sense because the wave function fits right in between two infinite potential barriers.

Now we make the potential barrier finite but large ($\kappa a \gg 1$), and two states must split. To see the difference between two energy levels, we note that the only difference is between $\tanh \kappa a$ and $\coth \kappa a$, and for a large $\kappa a \gg 1$, the difference is exponentially small, $\tanh \kappa a = 1 - 2 e^{-2\kappa a} + O(e^{-4\kappa a})$, $\coth \kappa a = 1 + 2 e^{-2\kappa a} + O(e^{-4\kappa a})$. Therefore we would like to sovle $\kappa a (1 \mp 2 e^{-2\kappa a}) = -k a \cot k b$ to find $k \simeq \frac{\pi}{b}$ for parities ± 1. We expand $k b = \pi - \epsilon$, and $-k a \cot k b = \frac{-k a \cos k b}{\sin k b} = \frac{k a}{\epsilon} + O(\epsilon) = \frac{\pi a}{b \epsilon} + O(\epsilon^0)$, and hence $\frac{\pi a}{b \epsilon} = \kappa a (1 \mp 2 e^{-2\kappa a})$, $\epsilon = \frac{\pi}{\kappa b}(1 \pm 2 e^{-2\kappa a})$. Therefore the energy eigenvalue is $E = \frac{\hbar^2 \pi^2}{2 m b^2}(1 - 2 \frac{1}{\kappa b}(1 \pm 2 e^{-2\kappa a}))$ to the leading order. The difference in the energies between the two low-lying states is exponentially suppressed as expected, $\Delta E = \frac{\hbar^2 \pi^2}{2 m b^2} \frac{8}{\kappa b} e^{-2\kappa a}$.

To plot the wave functions, we choose $a = b = 1$, $m = 1$, $\hbar = 1$, and $V_0 = 10$. Much larger V_0 makes it impossible to find the difference between the two energy eigenvalues numerically. We can solve numerically for k,

```
κ Tanh[κ a] == -k Cot[k b] /. {κ -> √(2 m V0 - k²)} /. {a → 1, b → 1, m → 1, ℏ → 1} /. {V0 → 10}
```

```
√(20 - k²) Tanh[√(20 - k²)] == -k Cot[k]
```

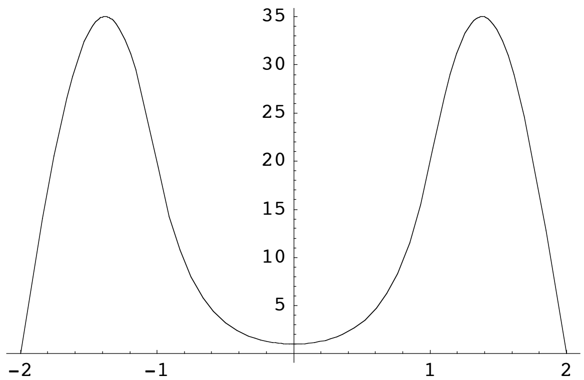
```
FindRoot[√(20 - k²) Tanh[√(20 - k²)] == -k Cot[k], {k, 3}]
```

```
{k → 2.53762}
```

```
ksol = k /. %
```

2.53762

$$\text{Plot}\left[\text{If}\left[\text{Abs}[x] < 1, \text{Cosh}\left[\sqrt{20 - \text{ksol}^2}\ x\right], \frac{\text{Cosh}\left[\sqrt{20 - \text{ksol}^2}\right]}{\text{Sin}[\text{ksol}\ (\text{Abs}[1] - 2)]}\ \text{Sin}[\text{ksol}\ (\text{Abs}[x] - 2)]\right], \{x, -2, 2\}\right]$$



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$$\varkappa\ \text{Coth}[\varkappa\ a] == -k\ \text{Cot}[k\ b]\ /.\ \left\{\varkappa \text{->} \sqrt{2\ m\ V_0 - k^2}\right\}\ /.\ \{a \to 1, b \to 1, m \to 1, \hbar \to 1\}\ /.\ \{V_0 \to 10\}$$

$$\sqrt{20 - k^2}\ \text{Coth}\left[\sqrt{20 - k^2}\right] == -k\ \text{Cot}[k]$$

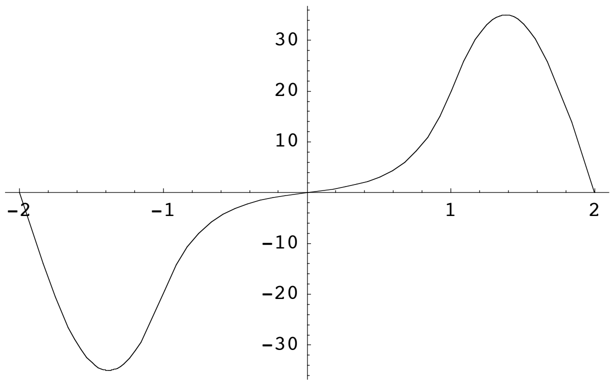
$$\text{FindRoot}\left[\sqrt{20 - k^2}\ \text{Coth}\left[\sqrt{20 - k^2}\right] == -k\ \text{Cot}[k], \{k, 3\}\right]$$

$\{k \to 2.53855\}$

```
ksol = k /. %
```

2.53855

```
Plot[If[Abs[x] < 1, Sinh[Sqrt[20 - ksol^2] x],
  Sign[x] Sinh[Sqrt[20 - ksol^2]] / Sin[ksol (Abs[1] - 2)] Sin[ksol (Abs[x] - 2)]], {x, -2, 2}]
```



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2. Periodic delta-function Potential

(a)

Using the form of the wave function given in the problem,
$\psi(-\epsilon) = A + B$
$\psi(+\epsilon) = e^{i k a}(A\, e^{-i \kappa a} + B\, e^{i \kappa a})$
$\psi'(-\epsilon) = i\, \kappa (A + B)$
$\psi'(+\epsilon) = i\, \kappa\, e^{i k a}(A\, e^{-i \kappa a} - B\, e^{i \kappa a})$
The wave function is continuous $\psi(-\epsilon) = \psi(+\epsilon)$, but its derivative is discontinuous because of the delta-function potential,
$\psi(+\epsilon) - \psi(-\epsilon) = \frac{2 m \lambda}{\hbar^2}\, \psi(0)$.

If you want to solve for "k" by hand the most expedient method is to write the two equations in the form of a matrix multiplying the column vector (A, B):

```
eqn1 = Collect[A + B - E^(I k a) (A E^(-I κ a) + B E^(I κ a)), {A, B}]
eqn2 = Collect[I κ E^(I k a) (A E^(-I κ a) - B E^(I κ a)) - I κ (A - B) - (2 m λ)/ħ^2 (A + B), {A, B}]
```

$$A\left(1 - e^{i a k - i a \kappa}\right) + B\left(1 - e^{i a k + i a \kappa}\right)$$

$$A\left(-i \kappa + i\, e^{i a k - i a \kappa} \kappa - \frac{2 m \lambda}{\hbar^2}\right) + B\left(i \kappa - i\, e^{i a k + i a \kappa} \kappa - \frac{2 m \lambda}{\hbar^2}\right)$$

```
myMatrix = {{Coefficient[eqn1, A], Coefficient[eqn1, B]},
   {Coefficient[eqn2, A], Coefficient[eqn2, B]}};
myMatrix // MatrixForm
```

$$\begin{pmatrix} 1 - e^{i a k - i a \kappa} & 1 - e^{i a k + i a \kappa} \\ -i\kappa + i e^{i a k - i a \kappa}\kappa - \frac{2 m \lambda}{\hbar^2} & i\kappa - i e^{i a k + i a \kappa}\kappa - \frac{2 m \lambda}{\hbar^2} \end{pmatrix}$$

The equation we want to solve is thus:

```
myMatrix.{A, B} == {0, 0}
```

$$\left\{A\,(1 - e^{i a k - i a \kappa}) + B\,(1 - e^{i a k + i a \kappa}),\ A\left(-i\kappa + i e^{i a k - i a \kappa}\kappa - \frac{2 m \lambda}{\hbar^2}\right) + B\left(i\kappa - i e^{i a k + i a \kappa}\kappa - \frac{2 m \lambda}{\hbar^2}\right)\right\} == \{0, 0\}$$

For there to be a nontrivial solution, the determinant of the coefficient matrix must be equal to zero. *Mathematica* is not convenient for doing this manipulation, but it isn't hard to calculate the 2x2 determinant by hand and set it equal to zero. It yields a quadratic equation for $e^{i k a}$ which can be easily solved using the quadratic equation to give the required result.

If you want to use *Mathematica* to arrive at the solution, here is an example method:

```
Solve[
 {A + B == E^(I k a) (A E^(-I κ a) + B E^(I κ a)), I κ E^(I k a) (A E^(-I κ a) - B E^(I κ a)) - I κ (A - B) == (2 m λ)/ℏ^2 (A + B)}, {B, k}]
```

```
Solve::ifun : Inverse functions are being used by Solve, so some
   solutions may not be found; use Reduce for complete solution information. More…
```

$$\Big\{\Big\{B \to \frac{1}{2 m \lambda}\Big(-A m\lambda - A e^{-2 i a \kappa} m\lambda - i A \kappa\hbar^2 + i A e^{-2 i a \kappa}\kappa\hbar^2 - i A e^{-2 i a \kappa}\sqrt{-4 e^{2 i a \kappa}\kappa^2\hbar^4 + (i m\lambda - i e^{2 i a \kappa} m\lambda + \kappa\hbar^2 + e^{2 i a \kappa}\kappa\hbar^2)^2}\Big),$$
$$k \to -\frac{1}{a}\Big(i\,\mathrm{Log}\Big[-\frac{1}{2\kappa\hbar^2}\Big(e^{-i a \kappa}\Big(-i m\lambda + i e^{2 i a \kappa} m\lambda - \kappa\hbar^2 - e^{2 i a \kappa}\kappa\hbar^2 + \sqrt{-4 e^{2 i a \kappa}\kappa^2\hbar^4 + (i m\lambda - i e^{2 i a \kappa} m\lambda + \kappa\hbar^2 + e^{2 i a \kappa}\kappa\hbar^2)^2}\Big)\Big)\Big]\Big)\Big\},$$
$$\Big\{B \to \frac{1}{2 m \lambda}\Big(-A m\lambda - A e^{-2 i a \kappa} m\lambda - i A \kappa\hbar^2 + i A e^{-2 i a \kappa}\kappa\hbar^2 + i A e^{-2 i a \kappa}\sqrt{-4 e^{2 i a \kappa}\kappa^2\hbar^4 + (i m\lambda - i e^{2 i a \kappa} m\lambda + \kappa\hbar^2 + e^{2 i a \kappa}\kappa\hbar^2)^2}\Big),$$
$$k \to -\frac{1}{a}\Big(i\,\mathrm{Log}\Big[-\frac{1}{2\kappa\hbar^2}\Big(e^{-i a \kappa}\Big(-i m\lambda + i e^{2 i a \kappa} m\lambda - \kappa\hbar^2 - e^{2 i a \kappa}\kappa\hbar^2 - \sqrt{-4 e^{2 i a \kappa}\kappa^2\hbar^4 + (i m\lambda - i e^{2 i a \kappa} m\lambda + \kappa\hbar^2 + e^{2 i a \kappa}\kappa\hbar^2)^2}\Big)\Big)\Big]\Big)\Big\}\Big\}$$

Therefore,

$$e^{i k a} = -\frac{1}{2\hbar^2\kappa} e^{-i\kappa a}\left(-i m\lambda(1 - e^{2 i\kappa a}) - \hbar^2\kappa(1 + e^{2 i\kappa a}) \pm \sqrt{-4\hbar^4\kappa^2 e^{2 i\kappa a} + (i m\lambda(1 - e^{2 i\kappa a}) + \hbar^2\kappa(1 + e^{2 i\kappa a}))^2}\right)$$

$$= -\frac{1}{2\hbar^2\kappa}\left(-m\lambda\sin\kappa a - \hbar^2\kappa\cos\kappa a \pm \sqrt{-4\hbar^4\kappa^2 + (2 m\lambda\sin\kappa a + 2\hbar^2\kappa\cos\kappa a)^2}\right)$$

$$= \cos\kappa a + \frac{m\lambda}{\hbar^2\kappa}\sin\kappa a \pm i\sqrt{1 - (\cos\kappa a + \frac{m\lambda}{\hbar^2\kappa}\sin\kappa a)^2}\,.$$

As suggested in the problem, we define $d = \frac{\hbar^2}{m\lambda}$, and the expression simplifies to

$$e^{i k a} = \cos\kappa a + \frac{1}{\kappa d}\sin\kappa a \pm i\sqrt{1 - (\cos\kappa a + \frac{1}{\kappa d}\sin\kappa a)^2}\ .$$

(b)

In the limit $d \to \infty$, which is nothing but a free particle without a potential, we have $e^{i k a} = \cos\kappa a \pm i\sqrt{1-\cos^2\kappa a} = e^{\pm i\kappa a}$, and hence $\kappa = \pm(k + \frac{2\pi n}{a})$. Or equivalently, k is the momentum modulo $\frac{2\pi n}{a}$. Therefore, κ and hence the energy grows continuously as a function of k. This can be seen with a large enough d numerically:

```
Plot[-I / a * Log[Cos[κ a] + 1/(κ d) Sin[κ a] + I * Sqrt[1 - (Cos[κ a] + 1/(κ d) Sin[κ a])^2]] /.
   {a → 1, d → 100} /. {κ → π x}, {x, 0, 6}, PlotRange → {-Pi, Pi}]
Plot[-I / a * Log[Cos[κ a] + 1/(κ d) Sin[κ a] - I * Sqrt[1 - (Cos[κ a] + 1/(κ d) Sin[κ a])^2]] /.
   {a → 1, d → 100} /. {κ → π x}, {x, 0, 6}, PlotRange → {-Pi, Pi}]
```

```
Plot::plnr : - (i Log[Cos[κ a] + Power[<<2>>] <<1>> + i Sqrt[Plus[<<2>>]]])/a /. {a → 1, d → 100} /. {κ → π x}
   is not a machine-size real number at x =  2.5`*^-7. More…

Plot::plnr : - (i Log[Cos[κ a] + Power[<<2>>] <<1>> + i Sqrt[Plus[<<2>>]]])/a /. {a → 1, d → 100} /. {κ → π x}
   is not a machine-size real number at x =  0.029276095942973535` . More…

Plot::plnr : - (i Log[Cos[κ a] + Power[<<2>>] <<1>> + i Sqrt[Plus[<<2>>]]])/a /. {a → 1, d → 100} /. {κ → π x}
   is not a machine-size real number at x =  0.04245341352018174` . More…

General::stop : Further output of  Plot::plnr will be suppressed during this calculation. \! \(More…\)
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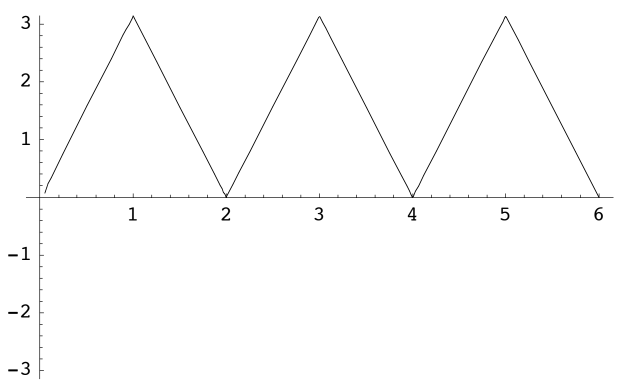

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```
Plot::plnr : - (i Log[Cos[κ a] + Power[<<2>>] <<1>> - i Power[<<2>>]])/a /. {a → 1, d → 100} /. {κ → π x}
   is not a machine-size real number at x = 2.5`*^-7.  More…

Plot::plnr : - (i Log[Cos[κ a] + Power[<<2>>] <<1>> - i Power[<<2>>]])/a /. {a → 1, d → 100} /. {κ → π x}
   is not a machine-size real number at x = 0.029276095942973535` . More…

Plot::plnr : - (i Log[Cos[κ a] + Power[<<2>>] <<1>> - i Power[<<2>>]])/a /. {a → 1, d → 100} /. {κ → π x}
   is not a machine-size real number at x =  0.04245341352018174` . More…
```

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General::stop : Further output of  Plot::plnr will be suppressed during this calculation. \! \(More…\)
```

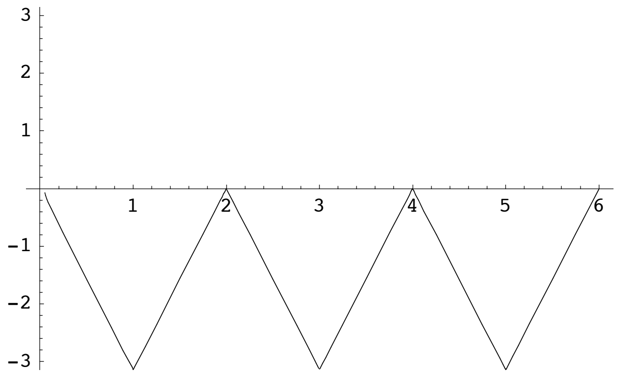

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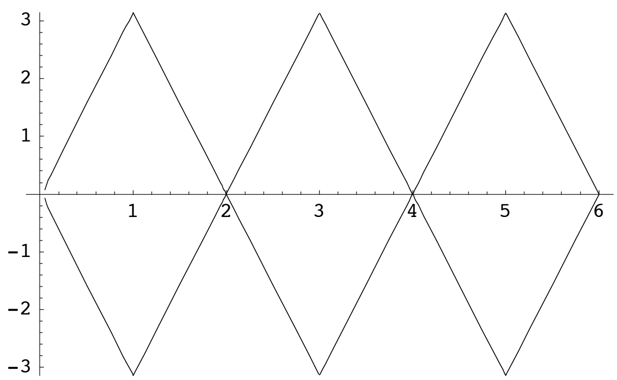

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(c)

Looking at the equation $e^{i k a} = \cos\kappa a + \frac{1}{\kappa d}\sin\kappa a \pm i\sqrt{1-(\cos\kappa a + \frac{1}{\kappa d}\sin\kappa a)^2}$, if the argument of the square root is negative, the l.h.s. becomes pure real and cannot satisfy the equation for a real k. Therefore there is no solution when $|\cos\kappa a + \frac{1}{\kappa d}\sin\kappa a| > 1$. When d is finite but large, the combination exceeds unity for $\kappa a = n\pi + \epsilon$ $(\epsilon > 0)$. This can be seen by expanding it in terms of ϵ, $\cos(n\pi+\epsilon) = (-1)^n\left(1-\frac{\epsilon^2}{2}+O(\epsilon^4)\right)$, $\sin(n\pi+\epsilon) = (-1)^n(\epsilon + O(\epsilon^3))$, and hence $\cos\kappa a + \frac{1}{\kappa d}\sin\kappa a = (-1)^n\left(1+\frac{1}{\kappa d}\epsilon - \frac{\epsilon^2}{2} + O(\epsilon^3)\right)$, and the magnitude exceeds unity for $0 < \epsilon < \frac{1}{2\kappa d} \simeq \frac{1}{2n\pi d}$. The gap must exist just above $\kappa = \frac{n\pi}{a}$ for any n, while the gap becomes smaller for large n.

(d)

First for a weak potential $d = 3$,

```
Plot[-I / a * Log[Cos[κ a] + 1/(κ d) Sin[κ a] + I * Sqrt[1 - (Cos[κ a] + 1/(κ d) Sin[κ a])^2]] /. {a → 1, d → 3} /.
  {κ → π x}, {x, 0, 4}, PlotRange → {-Pi, Pi}]
Plot[-I / a * Log[Cos[κ a] + 1/(κ d) Sin[κ a] - I * Sqrt[1 - (Cos[κ a] + 1/(κ d) Sin[κ a])^2]] /. {a → 1, d → 3} /.
  {κ → π x}, {x, 0, 4}, PlotRange → {-Pi, Pi}]
```

```
Plot::plnr : -(i Log[Cos[κ a] + Power[<<2>>] <<1>> + i Sqrt[Plus[<<2>>]]])/a /. {a → 1, d → 3} /. {κ → π x}
   is not a machine-size real number at x =  1.6666666666666665`*^-7 . More…

Plot::plnr : -(i Log[Cos[κ a] + Power[<<2>>] <<1>> + i Sqrt[Plus[<<2>>]]])/a /. {a → 1, d → 3} /. {κ → π x}
   is not a machine-size real number at x =  0.16226796629166315` . More…

Plot::plnr : -(i Log[Cos[κ a] + Power[<<2>>] <<1>> + i Sqrt[Plus[<<2>>]]])/a /. {a → 1, d → 3} /. {κ → π x}
   is not a machine-size real number at x =  0.24672633017238965` . More…

General::stop : Further output of  Plot::plnr will be suppressed during this calculation. \! \(More…\)
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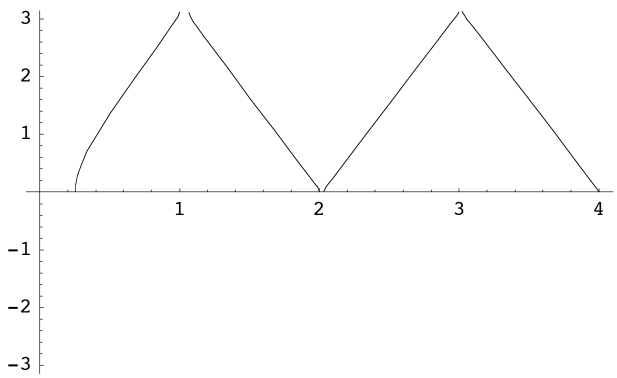

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```
Plot::plnr : -(i Log[Cos[κ a] + Power[<<2>>] <<1>> - i Power[<<2>>]])/a /. {a → 1, d → 3} /. {κ → π x}
   is not a machine-size real number at x =  1.6666666666666665`*^-7 . More…

Plot::plnr : -(i Log[Cos[κ a] + Power[<<2>>] <<1>> - i Power[<<2>>]])/a /. {a → 1, d → 3} /. {κ → π x}
   is not a machine-size real number at x =  0.16226796629166315` . More…

Plot::plnr : -(i Log[Cos[κ a] + Power[<<2>>] <<1>> - i Power[<<2>>]])/a /. {a → 1, d → 3} /. {κ → π x}
   is not a machine-size real number at x =  0.24672633017238965` . More…

General::stop : Further output of  Plot::plnr will be suppressed during this calculation. \! \(More…\)
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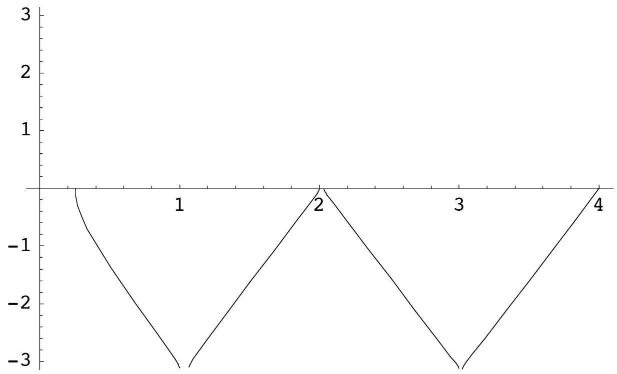

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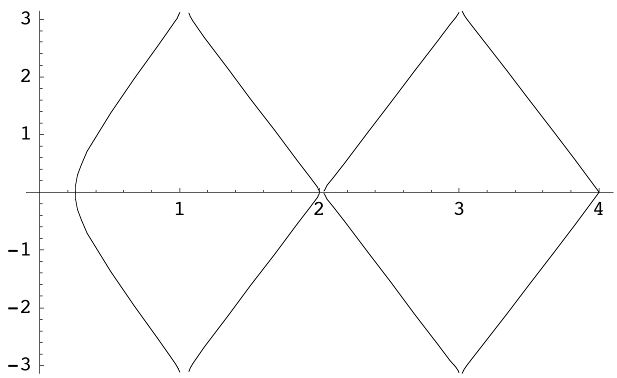

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There are gaps just above $k = \frac{n\pi}{a}$, and the gaps become smaller for higher n as expected from the analytic considerations in the part (c).

Now for a strong potential $d = \frac{1}{3}$,

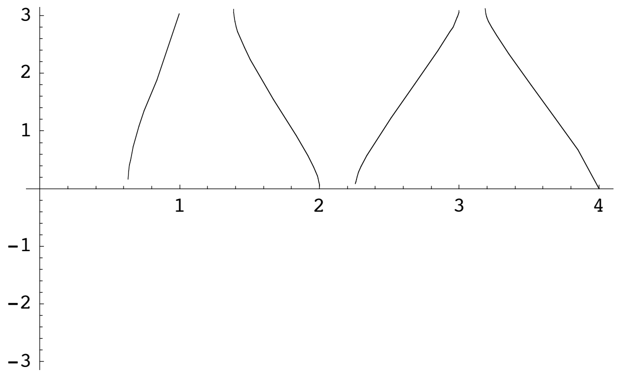
```
Plot[-I / a * Log[Cos[κ a] + 1/(κ d) Sin[κ a] + I * Sqrt[1 - (Cos[κ a] + 1/(κ d) Sin[κ a])^2]] /.
    {a → 1, d → 1 / 3} /. {κ → π x}, {x, 0, 4}, PlotRange → {-Pi, Pi}]
Plot[-I / a * Log[Cos[κ a] + 1/(κ d) Sin[κ a] - I * Sqrt[1 - (Cos[κ a] + 1/(κ d) Sin[κ a])^2]] /.
    {a → 1, d → 1 / 3} /. {κ → π x}, {x, 0, 4}, PlotRange → {-Pi, Pi}]
```

```
Plot::plnr : - (i Log[Cos[κ a] + Power[<<2>>] <<1>> + i √Plus[<<2>>]]) / a /. {a → 1, d → 1/3} /. {κ → π x}
   is not a machine-size real number at x = 1.6666666666666665`*^-7 . More…

Plot::plnr : - (i Log[Cos[κ a] + Power[<<2>>] <<1>> + i √Plus[<<2>>]]) / a /. {a → 1, d → 1/3} /. {κ → π x}
   is not a machine-size real number at x = 0.16226796629166315` . More…

Plot::plnr : - (i Log[Cos[κ a] + Power[<<2>>] <<1>> + i √Plus[<<2>>]]) / a /. {a → 1, d → 1/3} /. {κ → π x}
   is not a machine-size real number at x = 0.3392351994374947` . More…
```

General::stop : Further output of Plot::plnr will be suppressed during this calculation. \! \(More…\)



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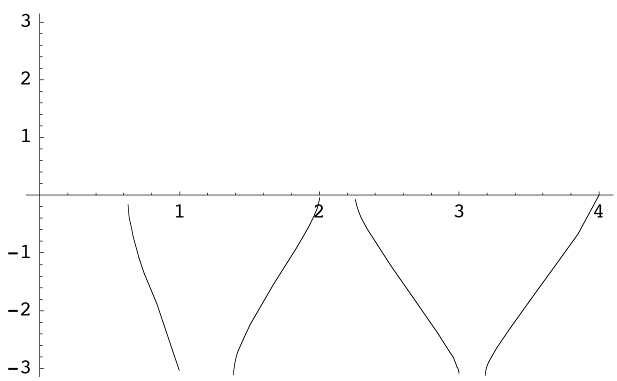
Plot::plnr : $-\frac{\text{i Log[Cos[}\kappa\text{ a] + Power[}\ll 2\gg\text{] }\ll 1\gg\text{ - i Power[}\ll 2\gg\text{]]}}{\text{a}}$ /. $\{a \to 1, d \to \frac{1}{3}\}$ /. $\{\kappa \to \pi x\}$
is not a machine-size real number at x = 1.6666666666666665`*^-7 . More…

Plot::plnr : $-\frac{\text{i Log[Cos[}\kappa\text{ a] + Power[}\ll 2\gg\text{] }\ll 1\gg\text{ - i Power[}\ll 2\gg\text{]]}}{\text{a}}$ /. $\{a \to 1, d \to \frac{1}{3}\}$ /. $\{\kappa \to \pi x\}$
is not a machine-size real number at x = 0.16226796629166315` . More…

Plot::plnr : $-\frac{\text{i Log[Cos[}\kappa\text{ a] + Power[}\ll 2\gg\text{] }\ll 1\gg\text{ - i Power[}\ll 2\gg\text{]]}}{\text{a}}$ /. $\{a \to 1, d \to \frac{1}{3}\}$ /. $\{\kappa \to \pi x\}$
is not a machine-size real number at x = 0.3392351994374947` . More…

General::stop : Further output of Plot::plnr will be suppressed during this calculation. \! \(More…\)



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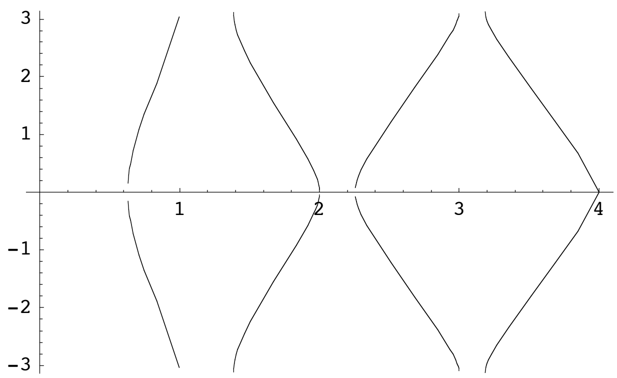

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The result is highly distorted from the free-particle case.
Nonetheless the band structure is clearly seen.

(e)

It is parity, that changes the overall sign of k. This can be seen from the explicit form of the wave function,

$\psi(x) = A\, e^{i\kappa x} + B\, e^{-i\kappa x}$ for $(-a < x < 0)$

$\psi(x) = e^{i k a}(A\, e^{i\kappa(x-a)} + B\, e^{-i\kappa(x-a)})$ for $0 < x < a$.

The parity transforms it to

$\psi(x) = e^{i k a}(A\, e^{i\kappa(-x-a)} + B\, e^{-i\kappa(-x-a)}) = B\, e^{i(k+\kappa)a}\, e^{i\kappa x} + A\, e^{i(k-\kappa)a}\, e^{-i\kappa x} = A'\, e^{i\kappa x} + B'\, e^{-i\kappa x}$ for $(-a < x < 0)$

$\psi(x) = B\, e^{i\kappa x} + A\, e^{-i\kappa x} = e^{-i k a}(B\, e^{i(k+\kappa)a}\, e^{i\kappa(x-a)} + A\, e^{i(k-\kappa)a}\, e^{-i\kappa(x-a)}) = e^{-i k a}(A'\, e^{i\kappa(x-a)} + B'\, e^{-i\kappa(x-a)})$ for $0 < x < a$.

The two wave functions are related by the change

$A \to A' = B\, e^{i(k+\kappa)a}$, $B \to A\, e^{i(k-\kappa)a}$, $e^{i k a} \to e^{-i k a}$.