Midterm

1. Particle on a circle

(a) Into the Lagrangian of a point particle $L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right)$, we substitute in $x = R\cos\theta$, $y = R\sin\theta$. Because the particle is always at the radius R, $\dot{x} = -R\,\dot{\theta}\sin\theta$, $\dot{y} = R\,\dot{\theta}\cos\theta$, and hence $L = \frac{1}{2} m R^2 \dot{\theta}^2$.

(b) The canonical momentum is given by its definition, $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m R^2 \dot{\theta}$. The Hamiltonian is

$$H = p_\theta \dot{\theta} - L = p_\theta \frac{p_\theta}{m R^2} - \frac{1}{2} m R^2 \left(\frac{p_\theta}{m R^2}\right)^2 = \frac{p_\theta{}^2}{2 m R^2}.$$

(c) The Heisenberg equation of motion is

$$i\hbar \frac{d}{dt}\theta = [\theta, H] = \left[\theta, \frac{p_\theta{}^2}{2 m R^2}\right] = i\hbar \frac{p_\theta}{m R^2}$$

$$i\hbar \frac{d}{dt} p_\theta = [p_\theta, H] = \left[p_\theta, \frac{p_\theta{}^2}{2m R^2}\right] = 0.$$

The solution to the second equation is simply that $p_\theta(t) = p_\theta(0)$ is conserved, and hence

$$\theta(t) = \theta(0) + \frac{p_\theta}{m R^2}\, t.$$

(d) The position-space wave function is $\langle\theta \mid p_\theta \mid k\rangle = \frac{\hbar}{i}\frac{\partial}{\partial\theta}\langle\theta \mid k\rangle = \hbar k \langle\theta \mid k\rangle$, and hence $\psi(\theta) = \langle\theta \mid k\rangle = N e^{i k \theta}$. To normalize it, we require $\int_0^{2\pi} |\psi(\theta)|^2 d\theta = 2\pi N^2 = 1$, and hence $N = 1\big/\sqrt{2\pi}$, $\psi(\theta) = \frac{1}{\sqrt{2\pi}} e^{i k\theta}$. In order to satisfy $\psi(\theta + 2\pi) = \psi(\theta)$, namely $e^{2\pi i k} = 1$, we need k to be an integer.

(e) The Schrödinger equation is

$$i\hbar \frac{d}{dt}\psi(\theta) = \langle\theta \mid i\hbar \frac{\partial}{\partial t} \mid \psi\rangle = \left\langle\theta \middle| \frac{p_\theta{}^2}{2m R^2} \middle| \psi\right\rangle = \frac{1}{2 m R^2}\left(\frac{\hbar}{i}\frac{\partial}{\partial\theta}\right)^2 \langle\theta \mid \psi\rangle = -\frac{\hbar^2}{2m R^2}\frac{\partial^2}{\partial\theta^2}\psi(\theta).$$

Its complex conjugate is

$$-i\hbar \frac{d}{dt}\psi^*(\theta) = -\frac{\hbar^2}{2 m R^2}\frac{\partial^2}{\partial\theta^2}\psi^*(\theta).$$

Using them, we find

$$\frac{\partial}{\partial t}\rho = \frac{\partial}{\partial t}\psi^*\psi = \psi^*\left(i\frac{\hbar}{2 m R^2}\frac{\partial^2}{\partial\theta^2}\psi\right) + \left(-i\frac{\hbar}{2 m R^2}\frac{\partial^2}{\partial\theta^2}\psi^*\right)\psi = \frac{\partial}{\partial\theta}\frac{i\hbar}{2 m R^2}\left(\psi^*\frac{\partial\psi}{\partial\theta} - \frac{\partial\psi^*}{\partial\theta}\psi\right) = -\frac{\partial j_\theta}{\partial\theta}.$$

For the state $|n\rangle$, we find

$$\rho = \frac{1}{2\pi}$$

is a constant for the entire circle, while

$$j_\theta = \frac{\hbar}{2 i m R^2}\left(\frac{e^{-i n\theta}}{\sqrt{2\pi}}\left(i n \frac{e^{i n\theta}}{\sqrt{2\pi}}\right) - \left(-i n \frac{e^{-i n\theta}}{\sqrt{2\pi}}\right)\frac{e^{i n\theta}}{\sqrt{2\pi}}\right) = \frac{n\hbar}{2\pi m R^2}.$$

Therefore the probability current is constant flow along the circle depending on the value of n.

(f) The orthonormality is simply

$$\langle n \mid m\rangle = \int_0^{2\pi}\langle n \mid \theta\rangle d\theta \langle\theta \mid m\rangle = \int_0^{2\pi}\frac{e^{-i n\theta}}{\sqrt{2\pi}}\frac{e^{i m\theta}}{\sqrt{2\pi}} d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta} d\theta.$$

When $n = m$, the integrand is unity, and hence $\langle n \mid n\rangle = \frac{1}{2\pi} 2\pi = 1$ and is normalized.
When $n \neq m$, $\langle n \mid m\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i(m-n)\theta} d\theta = \frac{1}{2\pi}\left[\frac{e^{i(m-n)\theta}}{i(m-n)}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{1}{i(m-n)} - \frac{1}{i(m-n)}\right] = 0$ and are orthogonal.

(g) With the vector potential, there is an additional term to the Lagrangian

$$L_{\rm int} = q\vec{A}\cdot\vec{v} = q\left(A_x \dot{x} + A_y \dot{y}\right) = q\frac{B}{2}\frac{d^2}{R^2}\left(y R\dot{\theta}\sin\theta + x R\dot{\theta}\cos\theta\right) = \frac{1}{2} q B d^2 \dot{\theta}.$$

Therefore, the total Lagrangian is

$$L = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} q B d^2 \dot{\theta},$$

and hence the canonical momentum is modified as

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = m R^2 \dot{\theta} + \frac{1}{2} q B d^2 .$$

The Hamitonian is therefore

$$H = p_\theta \dot{\theta} - L = p_\theta \frac{p_\theta - \frac{1}{2} qB\, d^2}{m R^2} - \frac{1}{2} m R^2 \left(\frac{p_\theta - \frac{1}{2} q B d^2}{m R^2}\right)^2 + \frac{1}{2} q B R^2 \frac{p_\theta - \frac{1}{2} q B d^2}{m R^2}$$

$$= \frac{1}{m R^2} \left(p_\theta{}^2 - \frac{1}{2} q B d^2 p_\theta - \frac{1}{2} p_\theta{}^2 + \frac{1}{2} q B d^2 p_\theta - \frac{1}{8} (q B d^2)^2 - \frac{1}{2} q B d^2 p_\theta + \frac{1}{4} (q B d^2)^2\right)$$

$$= \frac{1}{m R^2} \left(\frac{1}{2} p_\theta{}^2 - \frac{1}{2} q B d^2 p_\theta + \frac{1}{8} (q B d^2)^2 = \frac{1}{2 m R^2} (p_\theta - \frac{1}{2} q B d^2)^2 .\right.$$

The eigenvalues of the canonical momentum is still $p_\theta = n \hbar$ because of the periodicity requirement, and hence the energy eigenvalues are

$$E_n = \frac{1}{2 m R^2} (n \hbar - \frac{1}{2} q B d^2)^2 .$$

Even though the particle never "sees" the magnetic field, the energy eigenvalues are affected by the vector potential, another manifestation of the Aharonov-Bohm effect. Note also that the result depends only on the total magnetic flux $\frac{q \Phi}{2\hbar} = \frac{q B \pi d^2}{2 \pi \hbar}$ modulo integers.

2. Probability current

As we discussed in the class, the Hamitonian of a point particle in the presence of vector and scalar potentials is

$$H = \frac{1}{2m} \left(\vec{p} - q \vec{A}\right)^2 + q \phi.$$

The Schrödinger equation is therefore

$$i \hbar \frac{\partial}{\partial t} \psi = \left(\frac{1}{2 m} \left(\frac{\hbar}{i} \vec{\nabla} - q \vec{A}\right)^2 + q \phi\right) \psi,$$

and its complex conjugate is

$$-i \hbar \frac{\partial}{\partial t} \psi^* = \left(\frac{1}{2 m} \left(-\frac{\hbar}{i} \vec{\nabla} - q \vec{A}\right)^2 + q \phi\right) \psi^* ,$$

Therefore,

$$\frac{\partial}{\partial t} \rho = \frac{\partial}{\partial t} (\psi^* \psi) = \psi^* \frac{\partial \psi}{\partial t} + \frac{\partial \psi^*}{\partial t} \psi$$

$$= \psi^* \frac{1}{i \hbar} \left(\frac{1}{2 m} \left(\frac{\hbar}{i} \vec{\nabla} - q \vec{A}\right)^2 \psi + q \phi \psi\right) - \frac{1}{i \hbar} \left(\frac{1}{2m} \left(-\frac{\hbar}{i} \vec{\nabla} - q \vec{A}\right)^2 \psi^* + q \phi \psi^*\right) \psi$$

$$= \frac{i \hbar}{2 m} \left(\psi^* \left(\vec{\nabla}^2 \psi - i \frac{q}{\hbar} \vec{\nabla} \cdot \left(\vec{A} \psi\right) - i \frac{q}{\hbar} \vec{A} \cdot \vec{\nabla} \psi - \frac{q^2}{\hbar^2} \vec{A}^2 \psi\right)\right.$$

$$\left. - \left(\vec{\nabla}^2 \psi^* + i \frac{q}{\hbar} \vec{\nabla} \cdot \left(\vec{A} \psi^*\right) + i \frac{q}{\hbar} \vec{A} \cdot \vec{\nabla} \psi^* - \frac{q^2}{\hbar^2} \vec{A}^2 \psi^*\right) \psi\right)$$

$$= \frac{i \hbar}{2 m} \left(\psi^* \left(\vec{\nabla}^2 \psi\right) - \left(\vec{\nabla}^2 \psi^*\right) \psi - i \frac{q}{\hbar} \psi^* \vec{\nabla} \cdot \left(\vec{A} \psi\right) - i \frac{q}{\hbar} \vec{\nabla} \cdot \left(\vec{A} \psi^*\right) \psi - i \frac{q}{\hbar} \psi^* \vec{A} \cdot \vec{\nabla} \psi - \frac{q}{\hbar} \vec{A} \cdot \left(\vec{\nabla} \psi^*\right) \psi\right)$$

$$= \vec{\nabla} \cdot \frac{i \hbar}{2 m} \left(\psi^* \left(\vec{\nabla} \psi\right) - \left(\vec{\nabla} \psi^*\right) \psi - 2 i \frac{q}{\hbar} \psi^* \vec{A} \psi\right)$$

The conserved probability current is then

$$\vec{j} = \frac{\hbar}{2 i m} \left(\psi^* \left(\vec{\nabla} \psi\right) - \left(\vec{\nabla} \psi^*\right) \psi - 2 i \frac{q}{\hbar} \psi^* \vec{A} \psi\right)$$

$$= \frac{1}{2 m} \left(\psi^* \left(\frac{\hbar}{i} \vec{\nabla} - q \vec{A}\right) \psi + \left(\frac{-\hbar}{i} \vec{\nabla} - q \vec{A}\right) \psi^* \psi\right).$$

Under the gauge transformation, $\vec{A}' = \vec{A} - \vec{\nabla} \Lambda$, $\psi' = e^{-i q \Lambda / \hbar} \psi$, we can see

$$\left(\frac{\hbar}{i}\vec{\nabla}-q\vec{A}'\right)\psi' = \left(\frac{\hbar}{i}\vec{\nabla}-q\vec{A}+q\vec{\nabla}\Lambda\right)e^{-iq\Lambda/\hbar}\psi$$
$$= e^{-iq\Lambda/\hbar}\left(\frac{\hbar}{i}\vec{\nabla}-q\vec{\nabla}\Lambda-q\vec{A}+q\vec{\nabla}\Lambda\right)\psi = e^{-iq\Lambda/\hbar}\left(\frac{\hbar}{i}\vec{\nabla}-q\vec{A}\right)\psi.$$

Therefore, the probability current is transformed to

$$\vec{j}' = \frac{1}{2m}\left(e^{iq\Lambda/\hbar}\psi^* e^{-iq\Lambda/\hbar}\left(\frac{\hbar}{i}\vec{\nabla}-q\vec{A}\right)\psi + e^{iq\Lambda/\hbar}\left(\frac{-\hbar}{i}\vec{\nabla}-q\vec{A}\right)\psi^* e^{-iq\Lambda/\hbar}\psi\right) = \vec{j}$$

and hence is gauge invariant.

3. Landau levels

(a)

$[\Pi_x, \Pi_y] = [p_x - e A_x, p_y - e A_y] = -e[p_x, A_y] + e[p_y, A_x] = -e\frac{\hbar}{i}\partial_x A_y + e\frac{\hbar}{i}\partial_y A_x = i e\hbar B.$ Therefore, $[a, a^\dagger] = \frac{1}{2e\hbar B}[\Pi_x + i\Pi_y, \Pi_x - i\Pi_y] = \frac{1}{2e\hbar B}(-2i[\Pi_x, \Pi_y]) = \frac{1}{2e\hbar B}(-2 i i e\hbar B) = 1.$

(b)

Let us first work out $a^\dagger a = \frac{1}{2e\hbar B}(\Pi_x - i\Pi_y)(\Pi_x + i\Pi_y) = \frac{1}{2e\hbar B}(\Pi_x^2 + \Pi_y^2 + i[\Pi_x, \Pi_y]) = \frac{1}{2e\hbar B}(\Pi_x^2 + \Pi_y^2 - e\hbar B).$ Therefore,

$$H = \frac{1}{2m}(\Pi_x^2 + \Pi_y^2) = \frac{1}{2m}(2e\hbar B a^\dagger a + e\hbar B) = \frac{e\hbar B}{m}(a^\dagger a + \frac{1}{2}).$$

(c)

The ground state wave functions must satisfy $a|0\rangle = 0$ just like a harmonic oscillator. Note that $a = \frac{1}{\sqrt{2e\hbar B}}(p_x - e A_x + i(p_y - e A_y)) =$

$$\frac{1}{\sqrt{2e\hbar B}}\left(\frac{\hbar}{i}\partial_x + e\frac{B}{2}y + i\left(\frac{\hbar}{i}\partial_y - e\frac{B}{2}x\right)\right) = \frac{-i\hbar}{\sqrt{2e\hbar B}}\left((\partial_x + i\partial_y) + \frac{eB}{2\hbar}(x+iy)\right) = \frac{-i\hbar}{\sqrt{2e\hbar B}}\left(2\overline{\partial} + \frac{eB}{2\hbar}z\right),$$

where $\overline{\partial} = \frac{\partial}{\partial\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. Below, we also use the notation $\partial = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y)$ so that $\partial\bar{z} = \overline{\partial}z = 0$, $\partial z = \overline{\partial}\bar{z} = 1$. The ground state wave functions therefore satisfy

$$\left(\overline{\partial} + \frac{eB}{4\hbar}z\right)\psi(z,\bar{z}) = 0.$$

Writing $\psi(z,\bar{z}) = \psi'(z,\bar{z})\, e^{-eBz\bar{z}/4\hbar}$, we find

$$0 = \left(\overline{\partial} + \frac{eB}{4\hbar}z\right)\psi'(z,\bar{z})\, e^{-eBz\bar{z}/4\hbar} = (\overline{\partial}\psi'(z,\bar{z}))\, e^{-eBz\bar{z}/4\hbar}.$$

Therefore, any function $\psi'(z,\bar{z})$ that satisfies $\overline{\partial}\psi'(z,\bar{z}) = 0$, namely a function of z only with no dependence on $\bar{z}$ would satisfy the equation. In particular, $\psi' = z^n$ is a solution for any n.

```
Integrate[2 π r r^(2 n) E^(-e B r^2/(2 ħ)), {r, 0, ∞}, Assumptions -> Re[n] > -1 && Re[B e/ħ] > 0]
```

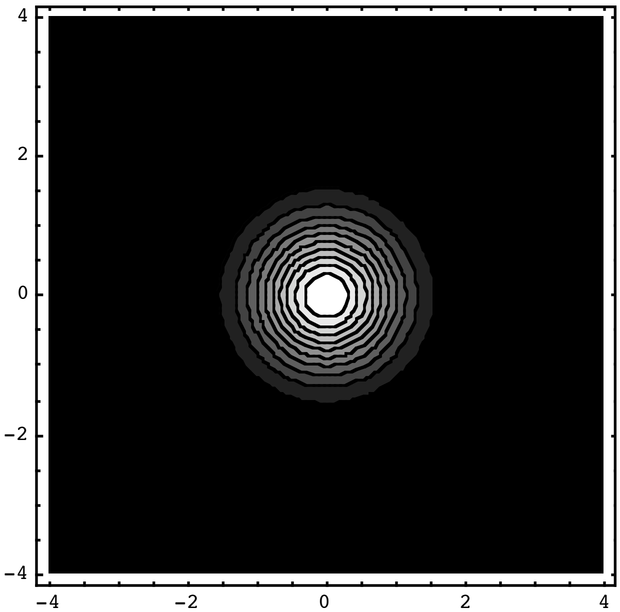
$$2^{1+n}\pi\left(\frac{Be}{\hbar}\right)^{-1-n}\text{Gamma}[1+n]$$

Therefore, $N = \left(n!\,\pi\left(\frac{2\hbar}{eB}\right)^{n+1}\right)^{-1/2}$.

(d)

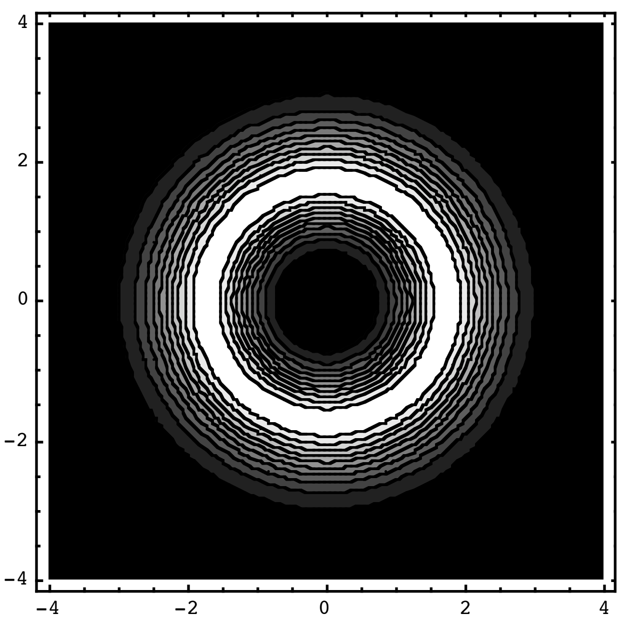
Take $eB/2\hbar = 1$. Then, $\psi_n = (n!\,\pi)^{-1/2} z^n e^{-\bar{z}z/2}$. Therefore,

```
In[15]:= ContourPlot[(n! π)^-1 r^(2n) E^(-r^2) /. {r -> √(x² + y²)} /. {n → 0},
         {x, -4, 4}, {y, -4, 4}, PlotPoints → 100, PlotRange → {0, 1/π}]
```



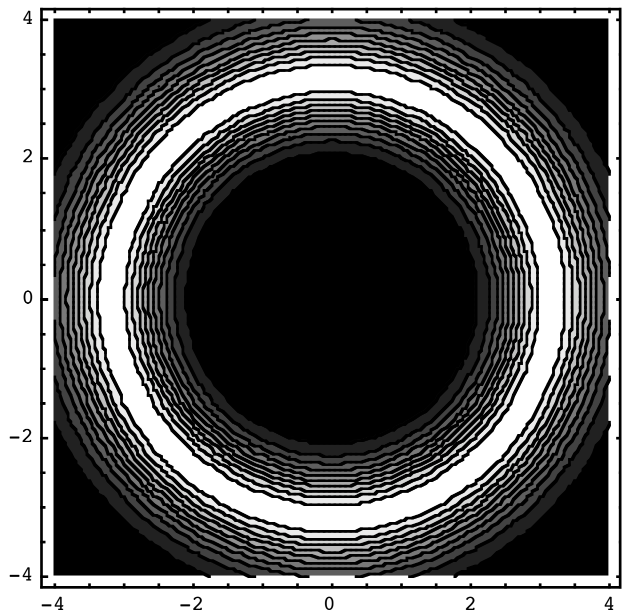
```
Out[15]= - ContourGraphics -
```

```
In[16]:= ContourPlot[(n! π)^-1 r^(2n) E^(-r^2) /. {r -> √(x² + y²)} /. {n → 3},
         {x, -4, 4}, {y, -4, 4}, PlotPoints → 100]
```



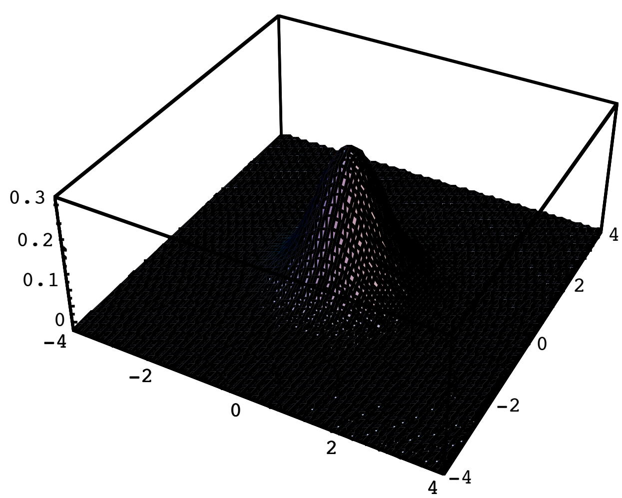
```
Out[16]= - ContourGraphics -
```

```
In[17]:= ContourPlot[(n! π)^-1 r^(2n) E^(-r^2) /. {r -> √(x² + y²)} /. {n → 10},
           {x, -4, 4}, {y, -4, 4}, PlotPoints → 100]
```



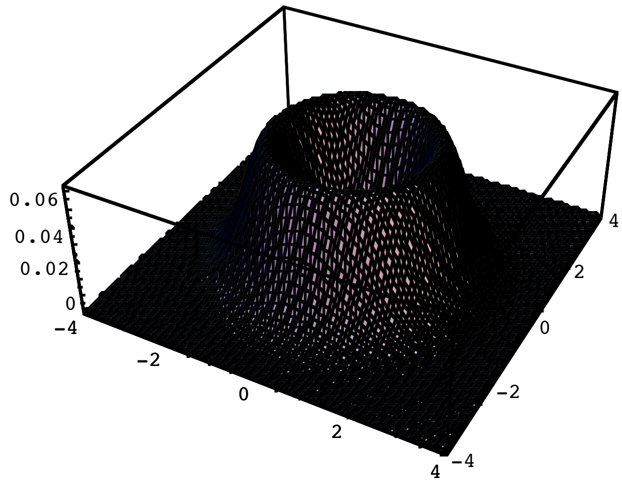

```
Out[17]=  - ContourGraphics -
```

```
In[11]:= Plot3D[(n! π)^-1 r^(2n) E^(-r^2) /. {r -> √(x² + y²)} /. {n → 0},
           {x, -4, 4}, {y, -4, 4}, PlotPoints → 100, PlotRange → {0, 1/π}]
```



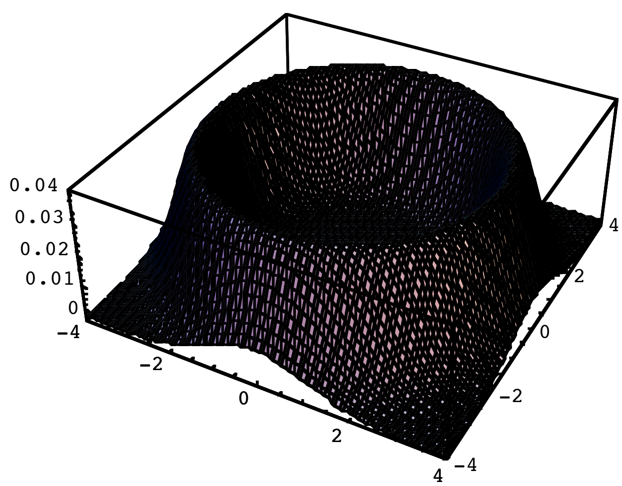

```
Out[11]=  - SurfaceGraphics -
```

```
In[12]:= Plot3D[(n! π)^-1 r^(2n) E^(-r^2) /. {r -> Sqrt[x^2 + y^2]} /. {n → 3},
           {x, -4, 4}, {y, -4, 4}, PlotPoints → 100]
```




```
Out[12]= - SurfaceGraphics -
```

```
In[13]:= Plot3D[(n! π)^-1 r^(2n) E^(-r^2) /. {r -> Sqrt[x^2 + y^2]} /. {n → 10},
           {x, -4, 4}, {y, -4, 4}, PlotPoints → 100]
```




```
Out[13]= - SurfaceGraphics -
```

(e)

The wave function forms a ring further and further away from the orign for larger and larger n. If the system has a finite radius R, the ring goes outside the system for too large n. This sets a maximum value on n, and hence there are only a finite number of ground states. To obtain the maximum n, we require that the peak of the probability density is less than R.

```
Solve[D[(r^n E^(-e B r^2/(4 ħ)))^2, r] == 0, r]
```

$$\left\{\left\{r\to-\frac{\sqrt{2}\,\sqrt{n}\,\sqrt{\hbar}}{\sqrt{B}\,\sqrt{e}}\right\},\ \left\{r\to\frac{\sqrt{2}\,\sqrt{n}\,\sqrt{\hbar}}{\sqrt{B}\,\sqrt{e}}\right\}\right\}$$

For this radius to be inside the system, $\frac{2n\hbar}{eB} < R^2$, and hence $n < \frac{eBR^2}{2\hbar}$. The number of ground states is therefore $\frac{1}{2\hbar}\, e B R^2$.

(f)

Using the expression $a = \frac{-i\hbar}{\sqrt{2e\hbar B}}(2\bar\partial + \frac{eB}{2\hbar}z)$, we find

$$a e^{-eB(z\bar z - 2z_0\bar z + z_0\bar z_0)/4\hbar} = \frac{-i\hbar}{\sqrt{2e\hbar B}}(2\bar\partial + \frac{eB}{2\hbar}z)\, e^{-eB(z\bar z - 2z_0\bar z + z_0\bar z_0)/4\hbar}$$

$$= \frac{-i\hbar}{\sqrt{2e\hbar B}}(-2\frac{eB}{4\hbar}(z - 2z_0) + \frac{eB}{2\hbar}z)\, e^{-eB(z\bar z - 2z_0\bar z + z_0\bar z_0)/4\hbar}$$

$$= \frac{-i\hbar}{\sqrt{2e\hbar B}}\frac{eB}{2\hbar}2z_0\, e^{-eB(z\bar z - 2z_0\bar z + z_0\bar z_0)/4\hbar}$$

$$= -i\sqrt{\frac{eB}{2\hbar}}\, z_0\, e^{-eB(z\bar z - 2z_0\bar z + z_0\bar z_0)/4\hbar}$$

Therefore, this wave function is an eigenstate of the annihilation operator with the eigenvalue $-i\sqrt{\frac{eB}{2\hbar}}\, z_0$, and can be regarded as a "coherent state" in analogy to the harmonic oscillator case.

The uncertainties of the wave function are calculated the usual way. The probability density is $|\psi|^2 = N_0^2 e^{-eB(2z\bar z - 2z_0\bar z - 2z\bar z_0 + 2z_0\bar z_0)/4\hbar} = N_0^2 e^{-eB(z - z_0)(\bar z - \bar z_0)/2\hbar}$ and hence is a Gaussian. Rewriting it in terms of Cartesian coordinates, $|\psi|^2 = N_0^2 e^{-eB((x-x_0)^2 + (y-y_0)^2)/2\hbar}$ for $z_0 = x_0 + i y_0$. Therefore, we find $\langle x\rangle = x_0$, $\langle y\rangle = y_0$, $(\Delta x)^2 = \langle (x - x_0)\rangle^2 = \frac{\hbar}{eB}$, $(\Delta y)^2 = \langle (y - y_0)\rangle^2 = \frac{\hbar}{eB}$. To calculate expectation values of the momentum, we first rewrite the wave function as $\psi = N_0 e^{-eB((x^2+y^2) - 2(x_0 + i y_0)(x - i y) + (x_0^2 + y_0^2))/4\hbar}$. $p_x\psi = \frac{\hbar}{i}\frac{-eB}{4\hbar}(2(x - x_0) - 2iy_0)\psi$. Therefore, $\langle p_x\rangle = \frac{\hbar}{i}\frac{-eB}{2\hbar}\langle x - x_0 - iy_0\rangle = \frac{\hbar}{i}\frac{-eB}{2\hbar}(-iy_0) = \frac{eB}{2}y_0$. Similarly, $p_y\psi = \frac{\hbar}{i}\frac{-eB}{4\hbar}(2(y - y_0) + 2ix_0)\psi$, and $\langle p_y\rangle = -\frac{eB}{2}x_0$. Finally, $\langle p_x^2\rangle = \int\int dx\, dy\, |p_x\psi|^2 = \int\int dx\, dy\, (\frac{eB}{2})^2((x - x_0)^2 + y_0^2)\left|\psi\right|^2 = (\frac{eB}{2})^2(\frac{\hbar}{eB} + y_0^2)$, and hence $(\Delta p_x)^2 = (\frac{eB}{2})^2\frac{\hbar}{eB}$. Similarly, $\langle p_y^2\rangle = \int\int dx\, dy\, |p_y\psi|^2 = \int\int dx\, dy\, (\frac{eB}{2})^2(x_0^2 + (y - y_0)^2)\left|\psi\right|^2 = (\frac{eB}{2})^2(x_0^2 + \frac{\hbar}{eB})$, and hence $(\Delta p_y)^2 = (\frac{eB}{2})^2\frac{\hbar}{eB}$ as well. The uncertainty relations are $(\Delta x)(\Delta p_x) = \hbar/2$, $(\Delta y)(\Delta p_y) = \hbar/2$, and this state is a minimunm uncertainty state.

acceptable solution

In the Heisenberg picture, we solve the equations of motion

$i\hbar\frac{d}{dt}x = [x, H] = \frac{i\hbar}{m}(p_x - eA_x) = \frac{i\hbar}{m}(p_x + \frac{eB}{2}y)$,

$i\hbar\frac{d}{dt}y = [y, H] = \frac{i\hbar}{m}(p_y - eA_y) = \frac{i\hbar}{m}(p_y - \frac{eB}{2}x)$,

$i\hbar\frac{d}{dt}p_x = [p_x, H] = \frac{-i\hbar}{m}(p_y - \frac{eB}{2}x)(-\frac{eB}{2})$,

$i\hbar\frac{d}{dt}p_y = [p_y, H] = \frac{-i\hbar}{m}(p_x + \frac{eB}{2}y)(\frac{eB}{2})$.

Taking another time derivative,

$\frac{d^2}{dt^2}x = \frac{d}{dt}\frac{1}{m}(p_x + \frac{eB}{2}y) = \frac{1}{m}(\frac{eB}{2m}(p_y - \frac{eB}{2}x) + \frac{eB}{2}\frac{\partial}{\partial t}y) = \frac{eB}{m}\frac{d}{dt}y$,

$\frac{d^2}{dt^2}y = \frac{1}{m}(-\frac{eB}{2m}(p_x + \frac{eB}{2}y) - \frac{eB}{2}\frac{d}{dt}x) = -\frac{eB}{m}\frac{d}{dt}x$.

Therefore, the solution is

$x(t) = x(0) + \frac{1}{\omega}\left(\dot x(0)\sin\omega t + \dot y(0)(1 - \cos\omega t)\right)$,

$y(t) = y(0) + \frac{1}{\omega}\left(-\dot x(0)(1 - \cos\omega t) + \dot y(0)\sin\omega t\right)$.

For our coherent state wave function, the initial values are given by

$\langle x(0)\rangle = x_0$, $\langle y(0)\rangle = y_0$, $\left\langle\dot x(0)\right\rangle = \frac{1}{m}\langle p_x + \frac{eB}{2}y\rangle = \omega y_0$, $\left\langle\dot y(0)\right\rangle = \frac{1}{m}\langle p_y - \frac{eB}{2}x\rangle = -\omega x_0$.

We find the expectation values as the function of time,

$\langle x(t)\rangle = x_0 + (y_0\sin\omega t - x_0(1 - \cos\omega t)) = x_0\cos\omega t + y_0\sin\omega t$,

$\langle y(t)\rangle = y_0 + (-y_0(1 - \cos\omega t) - x_0\sin\omega t) = -x_0\sin\omega t + y_0\cos\omega t$.

This shows that the expectation values undergo the classical cyclotron motion, as expected from the Ehrenfest theorem (Sakurai, p.87).

However, this calculation does not say anything about how the shape of the wave function evolves in time, if it comes back to the original one over time, etc. Any detailed information can be obtained by calculating $\langle x^2 \rangle$, $\langle x^3 \rangle$, etc, namely an infinite number of expectation values. But for that purpose the next approach would be better.

ideal solution with five bonus points

Using the expression $a^\dagger = \frac{-i\hbar}{\sqrt{2e\hbar B}}(2\partial - \frac{eB}{2\hbar}\bar{z})$, we can see $e^{-i\sqrt{eB/2\hbar c}\, z_0 a^\dagger} e^{-eBz\bar{z}/4\hbar} = e^{z_0(\partial - \frac{eB}{4\hbar}\bar{z})} e^{-eBz\bar{z}/4\hbar} = e^{z_0(-\frac{eB}{2\hbar}\bar{z})} e^{-eBz\bar{z}/4\hbar} = e^{-eB(z\bar{z}-2z_0\bar{z})/4\hbar}$, and this is the coherent state up to a constant. Under the time evolution, it changes to $e^{-iHt/\hbar} e^{-i\sqrt{eB/2\hbar c}\, z_0 a^\dagger} e^{-eBz\bar{z}/4\hbar} = e^{-iHt/\hbar} e^{-i\sqrt{eB/2\hbar c}\, z_0 a^\dagger} e^{iHt/\hbar} e^{-iHt/\hbar} e^{-eBz\bar{z}/4\hbar}$

$= e^{-i\sqrt{eB/2\hbar c}\, z_0 a^\dagger e^{-i\omega t}} e^{-i\omega t/2} e^{-eBz\bar{z}/4\hbar}$.

Here, we used the fact $e^{-iHt/\hbar} a^\dagger e^{iHt/\hbar} = a^\dagger e^{-i\omega t}$. Therefore, up to the overall phase factor due to the zero-point energy, the time evolution is simply a change of $z_0 \to z_0 e^{-i\omega t}$ in the wave function $e^{-eB(z\bar{z}-2z_0 e^{-i\omega t}\bar{z}+z_0\bar{z}_0)/4\hbar}$. It is always a Gaussian of the same form, but the center moves as $z = z_0 e^{-i\omega t}$, or $x = x_0 \cos\omega t + y_0 \sin\omega t$, $y = -x_0 \sin\omega t + y_0 \cos\omega t$, rotating clockwise on the complex plane of z around the origin.

One can see the time-dependence of the wave function by plotting the probability density over time: $|\psi|^2 = N^2 e^{-eB((x-x_0)^2+(y-y_0)^2)/2\hbar}$ with $x_0 = \mathrm{Re}(z_0 e^{-i\omega t})$, $y_0 = \mathrm{Im}(z_0 e^{-i\omega t})$. Take $eB = \hbar = 1$, $\omega = m^{-1} = 2\pi$, $z_0 = 5$. Execute the following command, select the following plots together, and go to "Cell" and "Animate Selected Graphics")

```
In[18]:= Table[ContourPlot[1/(2 π) Exp[-((x - 5 Cos[2 π t])^2 + (y + 5 Sin[2 π t])^2) / 2], {x, -10, 10},
    {y, -10, 10}, PlotPoints → 50, PlotRange → {0, 1/(2 π)}], {t, 0, 0.9, 0.1}]
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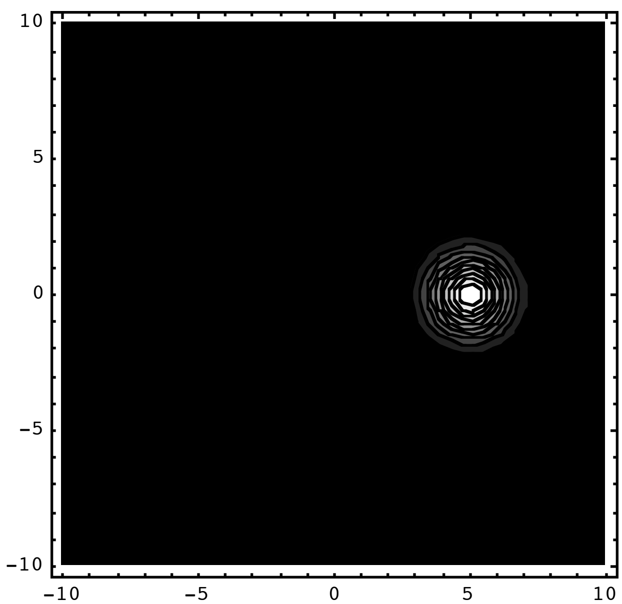

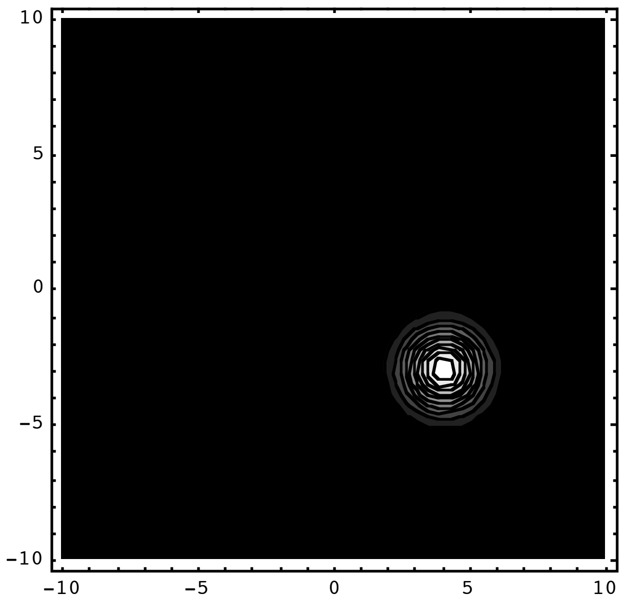
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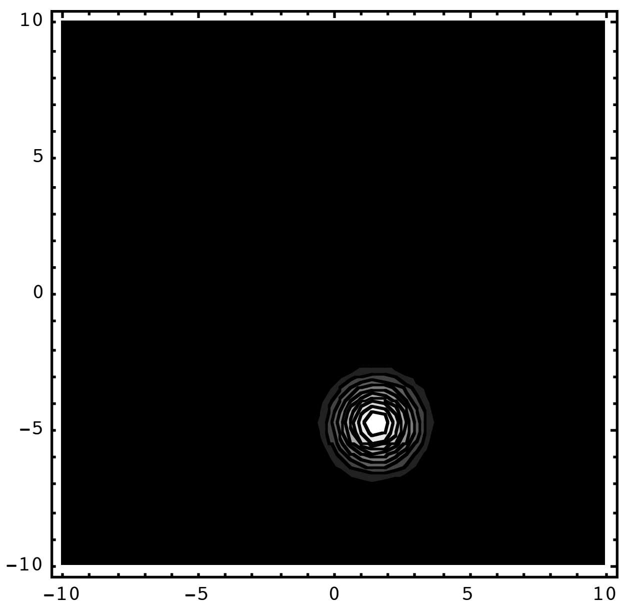
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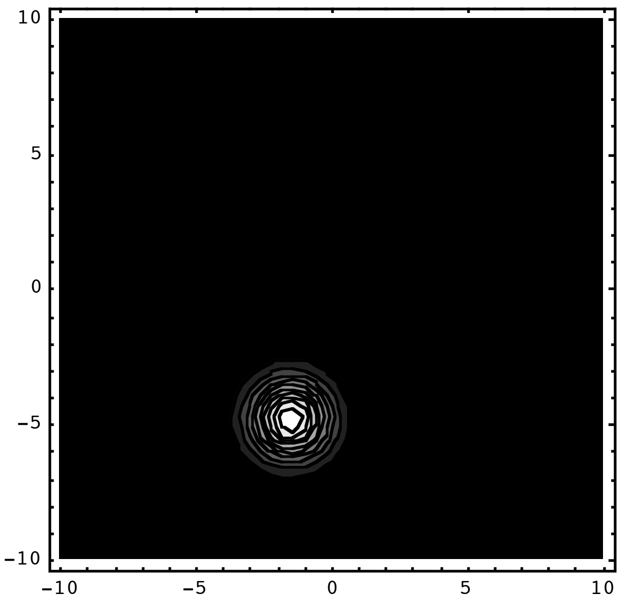
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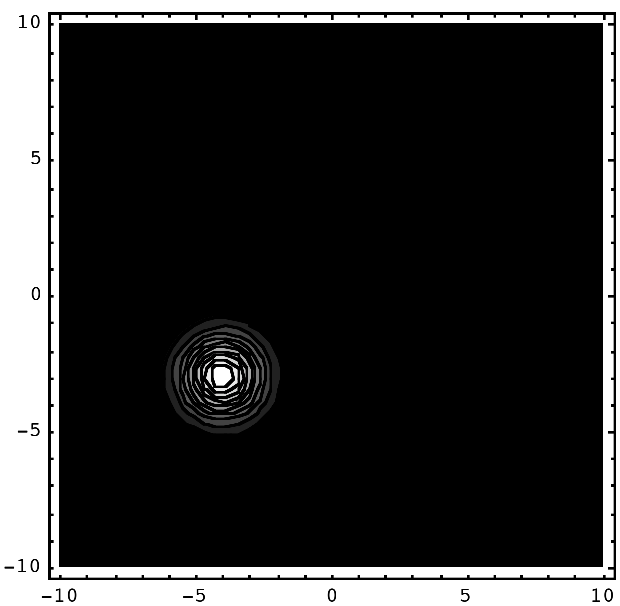
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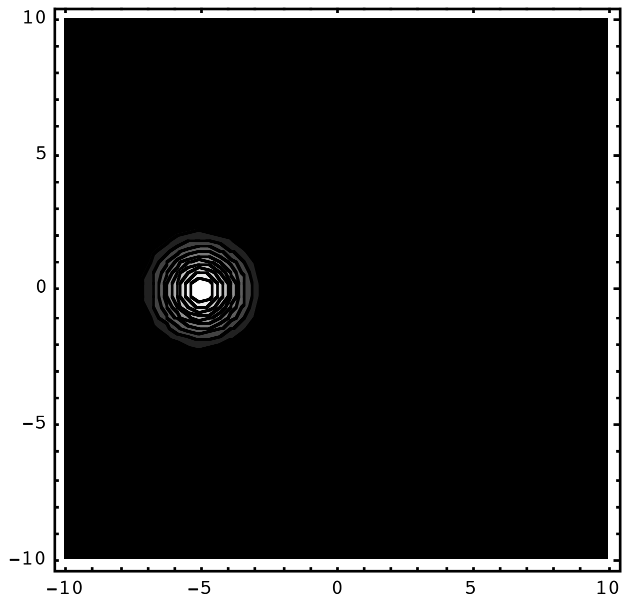

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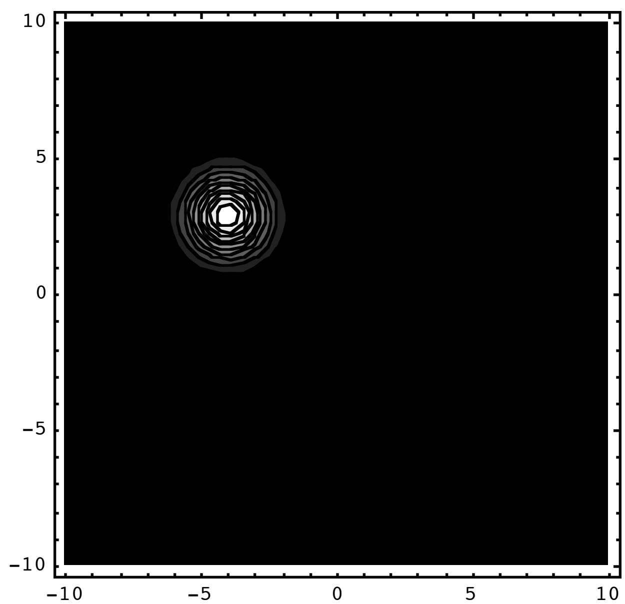

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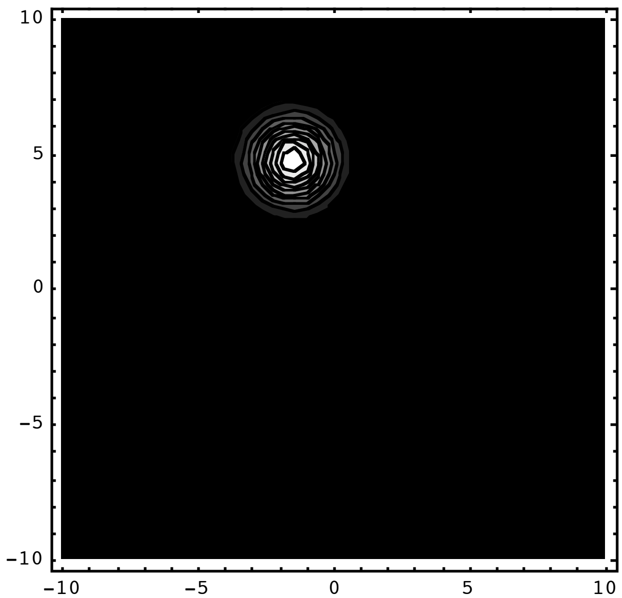
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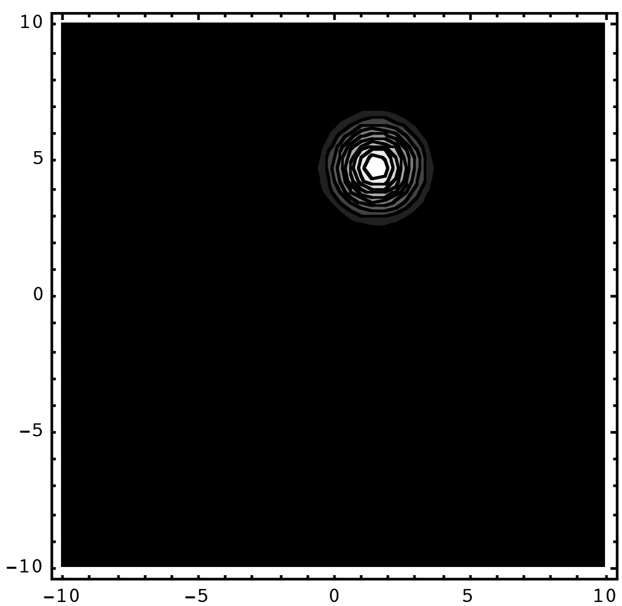
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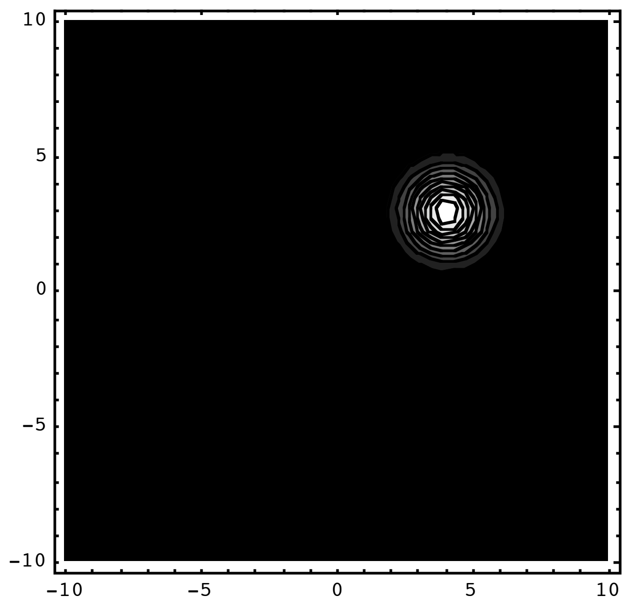


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(g)

For an electron, the "Physical Constants" table from the PDG says the Bohr magneton is $\mu_B = \frac{e\hbar}{2m_e} = 5.79\times 10^{-11}$ MeV T^{-1}. The excitation energy is $\Delta E = \frac{e\hbar B}{m_e} = 1.16\times 10^{-3}$ eV for $B = 100$ kG $= 10\,T$. The corresponding thermal energy is $\Delta E/k = 1.16\times 10^{-3}$ eV $/\, 8.62\times 10^{-5}$ eV $K^{-1} = 13.4\,K$. At temperatures below a few kelvin, pratically all electrons populate the ground states.

4. Scalar Aharonov-Bohm

In this experiment, a magnetic field is applied for $\Delta t = 8\,\mu$sec on neutrons whose magnetic moment is $\mu = -1.91\,\mu_N = -1.91\times 3.15\; 10^{-14}$ MeV $T^{-1} = -6.02\times 10^{-14}$ MeV T^{-1}. The relative phase between two waves is (following Eq. (3)),

$\Delta\Phi_{\rm AB} = \frac{1}{\hbar}\,\mu\, B\Delta t = \frac{1}{6.58\times 10^{-22}\ \text{MeV sec}}\,(-6.02\times 10^{-14}\ \text{MeV}\,T^{-1})\,B\times 8\times 10^{-6}\ \text{sec} = 7.32\times 10^{2}\ (B/T) = 0.0732\,(B/\text{Gauss})$.
The fit to the data shown in Fig. 5 says $\Delta\Phi_{\rm AB}/B = 0.0657/\text{Gauss}$, quite a good agreement with the expectation.

Fig. 2.4 in Sakurai's shows an electric potential, which has an electric field at the edges and hence forces. The reason why they chose to turn on and off the magnetic field is to avoid the possible criticism that a specially non-uniform field gives a non-uniform potential and hence a classical force. The purpose of the experiment, on the other hand, is to demonstrate the quantum phase in the absence of any classical force. Furthermore, they wanted to avoid the torque acting on the neutron spin, and therefore polarized the spins along the direction of the motion which is parallel to the magnetic field ("longitudinal polarization"). This way, they were sure that there is absolutely no classical force acting on neutrons, yet they showed the quantum phase, the scalar Aharonov-Bohm effect.