Midterm Exam (221A), due Oct 22, 4pm

1. A particle of mass m is allowed to move only along the circle of radius R on a plane, $x = R\cos\theta$, $y = R\sin\theta$.* [40]

   (a) Show that the Lagrangian is $L = \frac{m}{2}R^2\dot{\theta}^2$. Hereafter, θ is the generalized coordinate of this system. [5]

   (b) Write down the canonical momentum p_θ and the Hamiltonian. [5]

   (c) Write down the Heisenberg equation of motion, and solve them. (So far no representation was taken.) [5]

   (d) Write down the normalized position-space wave function for the momentum eigenstates $p_\theta|k\rangle = k\hbar|k\rangle$, and show that only $k = n \in \mathbb{Z}$ are allowed because of the requirement that $\psi(\theta + 2\pi) = \psi(\theta)$. [5]

   (e) Write down the Schrödinger equation in the position space, and that the following continuity equation for the probability holds

   $$\frac{\partial \rho}{\partial t} + \frac{\partial j_\theta}{\partial \theta} = 0, \qquad \rho = \psi^*\psi, \qquad j_\theta = \frac{i\hbar}{2mR^2}\left(\psi^*\frac{\partial \psi}{\partial \theta} - \frac{\partial \psi^*}{\partial \theta}\psi\right). \quad (1)$$

   Calculate ρ and j_θ for the state $|n\rangle$ and show that it has a constant flow of probability. [5]

   (f) Show the orthonormality $\langle n|m\rangle = \delta_{n,m}$ explicitly in the position space. [5]

   (g) Now we introduce a constant magnetic field B inside the radius $r < d < R$ but no magnetic field outside $r > d$, so that the vector potential is

   $$(A_x, A_y) = \begin{cases} \frac{B}{2}(-y, x) & (r < d) \\ \frac{B}{2}\frac{d^2}{r^2}(-y, x) & (r > d). \end{cases} \quad (2)$$

   Write the Lagrangian, derive the Hamiltonian, and show that energy eigenvalues are indeed influenced by the magnetic field even though the particle does not "see" the magnetic field directly. [10]

2. Write down the Schrödinger equation in the presence of vector and scalar potentials. Show that there is a conserved probability current that satisfies Sakurai's (2.4.15), but its form is modified from that in (2.4.16). Further show that it is gauge invariant. [10]

*This is a model of spacetime that has an extra spatial dimension curled up in small size. The energy levels in this problem would appear as the spectrum of masses (*i.e.*, rest energies) of a so-called Kaluza–Klein tower of particles, in honor of people who first speculated about an extra spatial dimension to our universe to unify the gravity and the electromagnetism.

3. Consider a charged particle on the x-y plane in a constant magnetic field perpendicular to the plane $\vec{B} = (0, 0, B)$ with the Hamiltonian[†] [40]

$$H = \frac{\Pi_x^2 + \Pi_y^2}{2m}, \qquad \Pi_i = p_i - eA_i. \tag{3}$$

   (a) Show that $[\Pi_x, \Pi_y] = ie\hbar B$, and hence that the "annihilation" operator $a = (\Pi_x + i\Pi_y)/\sqrt{2e\hbar B}$ satisfies the commutation relation $[a, a^\dagger] = 1$.

   (b) Rewrite the Hamiltonian using a and $a^\dagger$. [5]

   (c) Use the so-called "symmetric gauge" $\vec{A} = \frac{B}{2}(-y, x)$. Write down the differential equation for the ground state wave function in the position representation. Show that it is satisfied for

$$\psi_n(z, \bar{z}) = N z^n e^{-eB\bar{z}z/4\hbar}, \tag{4}$$

   where $z = x + iy$, $\bar{z} = x - iy$, and n is an arbitrary integer. Therefore the ground state is not unique and is highly degenerate. Determine the normalization factor N. [5]

   (d) Plot the probability density of the wave function for $n = 0$, 3, and 10 (use `ContourPlot` or `Plot3D`) on the same scale. [5]

   (e) Assuming that the system is a circle of a finite radius R, show that there are only a finite number of ground states. Work out the number. [5]

   (f) Show that the state

$$\psi(z, \bar{z}) = N e^{-eB(z-z_0)(\bar{z}-\bar{z}_0)/4\hbar} \tag{5}$$

   can be regarded as a "coherent state," has the minimum uncertainty, and that it undergoes a near-classical cyclotron motion. [10]

   (g) For an electron, what is the excitation energy under $B = 100$ kG? [5]

4. Read the article W.-T. Lee *et al*, "Observation of Scalar Ahoronov–Bohm Effect with Longitudinally Polarized Neutrons," *Phys. Rev. Lett.* **80**, 3165 (1998), which realized the gedanken experiment Sakurai discusses in pp. 123–125. Show that the expected $\Delta\Phi_{AB}$ in Eq. (3) depends on the magnetic field as is shown in Fig. 5. The magnetic moment of the neutron can be found from Particle Data Group at Lawrence Berkeley Laboratory. [10]

[†]This is the setup to study the Fractional Quantum Hall Effect. GaAs provides a plane where an electron is confined in a potential well along the z-direction, and can move only in two dimensions. A strong magnetic field of more than 100kG is then applied at temperatures below 1 K.