HW #7 (221A), due Oct 13, 4pm

1. Consider a free particle with Hamiltonian $H = \vec{p}^2/2m$.

   (a) Derive the propagator (Sakurai's Eq. (2.5.16))

   $$\langle \vec{x}'', t'' | \vec{x}' t' \rangle = \left(\frac{m}{2\pi i \hbar (t'' - t')} \right)^{3/2} \exp \left[\frac{im(\vec{x}'' - \vec{x}')^2}{2\hbar(t'' - t')} \right]. \tag{1}$$

   (b) Show that the exponent is nothing but the classical action for the particle to go from $\vec{x}'$ at t' to $\vec{x}''$ at t''.

   (c) Calculate the partition function of a free particle from Eq. (1).

   (d) Estimate the temperature for which the interchange of two identical particles would not cost to the Euclidean action

   $$S_E = \oint_0^{\beta\hbar} d\tau \frac{m}{2} \left(\frac{d\vec{x}}{d\tau} \right)^2 \tag{2}$$

   more than $\Delta S_E \simeq \hbar$ for liquid Helium.

2. The propagator contains full information about the system. Using the propagator for the harmonic oscillator, we can obtain energy levels and the wave functions in the following way.

   (a) Show that the propagator for imaginary time $(t_f - t_i) = -i\tau$ is in general given by

   $$K = \sum_n \psi_n(x_f)^* \psi_n(x_i) e^{-E_n \tau/\hbar}. \tag{3}$$

   (b) The propagator of harmonic oscillator (Eq. (2.5.18) in Sakurai) depends on τ only as $\sinh \omega\tau$ and $\cosh \omega\tau$, and hence can be rewritten as a function of $\epsilon = e^{-\omega\tau}$. Then identify the leading behavior when $\tau \to \infty$ and show that it oes as $\epsilon^{1/2} = e^{-\omega\tau/2}$. This way, we find that the ground state energy is $\frac{1}{2}\hbar\omega$.

   (c) Because we can expand the propagator in power series in ϵ as $\epsilon^{n+1/2}$, it is clear that the energy eigenvalues are $E_n = (n + \frac{1}{2})\hbar\omega$. The coefficient of $\epsilon^{n+1/2}$ s then the wave function $\psi_n(x_f)^*\psi_n(x_i)$. Work out the wave functions for $n = 0$, $n = 5$, and $n = 10$ this way. (It is quite impressive that the coefficient of $\epsilon^{n+1/2}$ always factorizes into a function of x_f and the same function of x_i.)

(d) Plot the obtained wavefunctions and verify that they are already properly normalized.

3. (optional) Work out the imaginary-time path integral for harmonic oscillator explicitly, for the interval $\tau = i(t_f - t_i)$ and discrete steps $\Delta\tau = \tau/N$. We will take the limit $N \to \infty$ in the end. The integral is given by

$$K(x_f, x_i, \tau) = \langle x_f | e^{-H\tau/\hbar} | x_i \rangle = \left(\frac{m}{2\pi\hbar\Delta\tau} \right)^{N/2} \int \prod_{n=1}^{N-1} dx_n e^{-S/\hbar}, \qquad (4)$$

where the discretized action is

$$S = \frac{m}{2} \sum_{n=0}^{N-1} \frac{(x_{n+1} - x_n)^2}{\Delta\tau} + \frac{m}{2}\omega^2 \left(\frac{1}{2}x_0^2 + \sum_{n=1}^{N-1} x_n^2 + \frac{1}{2}x_N^2 \right) \Delta\tau \qquad (5)$$

Here, $x_N = x_f$, $x_0 = x_i$. Therefore, the path integral is nothing but a big collection of Gaussian integrals because the Lagrangian of a harmonic oscillator is purely qudaratic.

To carry out the integral, a few identities would be useful. For $N-1$-dimensional column vectors, x, y, and a symmetric matrix A,

$$\int \prod_{n=1}^{N-1} dx_n e^{-\frac{1}{2}x^T A x - x^T y} = (2\pi)^{(N-1)/2} (\det A)^{-1/2} e^{+\frac{1}{2} y^T A^{-1} y}. \qquad (6)$$

For a $(N-1) \times (N-1)$ matrix of the form

$$A = \begin{pmatrix} 1 & -a & 0 & \cdots & 0 & 0 \\ -a & 1 & -a & \cdots & 0 & 0 \\ 0 & -a & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -a \\ 0 & 0 & 0 & \cdots & -a & 1 \end{pmatrix}, \qquad (7)$$

one can show

$$\det A = \frac{\lambda_+^N - \lambda_-^N}{\lambda_+ - \lambda_-}, \qquad \lambda_\pm = \frac{1}{2}(1 \pm \sqrt{1 - 4a^2}). \qquad (8)$$

Also,

$$\lim_{N\to\infty} \left(1 + \frac{x}{N} \right)^N = e^x. \qquad (9)$$