221A Lecture Notes
Supplemental Material on Harmonic Oscillator

1 Number-Phase Uncertainty

To discuss the harmonic oscillator with the Hamiltonian

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2, \tag{1}$$

we have defined the annihilation operator

$$a = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right), \tag{2}$$

the creation operator $a^\dagger$, and the number operator $N = a^\dagger a$.

In some discussions, it is useful to define the "phase" operator Θ by

$$a = e^{i\Theta}\sqrt{N}, \qquad a^\dagger = \sqrt{N}e^{-i\Theta}. \tag{3}$$

Obviously the phase is ill-defined when $N = 0$, but apart from that, it is a useful notion. It is particularly useful when we discuss the classical limit $N \gg 1$.

One can define the "phase eigenstate"

$$|\theta\rangle = \sum_{n=1}^{\infty} e^{in\theta}|n\rangle. \tag{4}$$

By acting the phase operator $e^{i\Theta} = a\frac{1}{\sqrt{N}}$,

$$
\begin{aligned}
e^{i\Theta}|\theta\rangle &= a\frac{1}{\sqrt{N}}\sum_{n=1}^{\infty} e^{in\theta}|n\rangle = \sum_{n=1}^{\infty} e^{in\theta}|n-1\rangle \\
&= \sum_{m=0}^{\infty} e^{i(m+1)\theta}|m\rangle = |0\rangle + e^{i\theta}|\theta\rangle.
\end{aligned}
\tag{5}
$$

It is *almost* an eigenstate of the phase operator, the failure due to the obvious problem with $n = 0$ state as anticipated from its definition. We can also calculate the inner products

$$\langle\theta'|\theta\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\langle m|e^{-im\theta'}e^{in\theta}|n\rangle = \sum_{n=1}^{\infty} e^{in(\theta-\theta')}. \tag{6}$$

This is *almost* the delta function

$$\delta(\theta - \theta') = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{in(\theta - \theta')}. \tag{7}$$

The number eigenstate is expressed correspondingly as

$$|n\rangle = \int_0^{2\pi} d\theta e^{-in\theta} |\theta\rangle, \tag{8}$$

which works for all n except for $n = 0$.

Again ignoring the subtlety with the $n = 0$ state, we can derive the number-phase uncertainty principle. Study the commutator

$$[N, e^{i\Theta}] = [N, a\frac{1}{\sqrt{N}}] = [N, a]\frac{1}{\sqrt{N}} = -a\frac{1}{\sqrt{N}} = -e^{i\Theta}. \tag{9}$$

Therefore, roughly speaking,

$$N = i\frac{\partial}{\partial \Theta}. \tag{10}$$

Indeed, this makes sense on the "phase eigenstate,"

$$\langle\theta|N = \sum_{n=1}^{\infty} e^{-in\theta}\langle n|n = i\frac{\partial}{\partial \theta}\langle\theta|. \tag{11}$$

Therefore, it leads to the "canonical commutation relation"

$$[N, \Theta] = i, \tag{12}$$

leading to the uncertainty principle

$$\Delta N \Delta \Theta \geq \frac{1}{2}. \tag{13}$$

2 Coherent State of Harmonic Oscillator

I've expanded discussions on the coherent state beyond Sakurai. Here is my lecture note on this subject.

We saw that the uncertainty of the state $|k\rangle$ is actually *larger* than the minimum uncertainty

$$\Delta x \Delta p = \frac{\hbar}{2}(2k + 1). \tag{14}$$

2

It appears odd that states with larger k, which we expect to behave more classicaly, are more uncertain. Moreover, expectation values of x and p vanish for energy eigenstates

$$\langle k|x|k \rangle = 0, \qquad \langle k|p|k \rangle = 0. \tag{15}$$

Therefore even for large k, the energy eigenstates do not share characteristics we expect for classical oscillators.

But how do we make a classical oscillator actually oscillate? Let's say we are talking about a pendulum. To make it oscillate, what we do is to exert a force on it, pull the pendulum up, make sure the pendulum is settled in your hand, and release it. Namely, pull, hold, and release. Why not do the same in quantum mechanics?

To pull a pendulum, we have to add an additional term to the potential

$$V = \frac{1}{2}m\omega^2 x^2 - Fx, \tag{16}$$

where F is the force we exert on the pendulum. Because the added term is linear in x, we can complete the square

$$V = \frac{1}{2}m\omega^2 (x - x_0)^2 - \frac{1}{2}m\omega^2 x_0^2, \tag{17}$$

so that the pendulum settles to the position $x_0 \neq 0$. The force for this purpose is given by $F = m\omega^2 x_0$. Because the pulled pendulum still has a quadratic potential, it is a modified harmonic oscillator. It settles to a ground state $|0\rangle'$, which is annihilated by the modified annihilation operator

$$a' = \sqrt{\frac{m\omega}{2\hbar}}\left((x - x_0) + \frac{ip}{m\omega}\right) = a - \sqrt{\frac{m\omega}{2\hbar}}x_0. \tag{18}$$

Therefore, the new ground state satisfies the equation

$$0 = a'|0\rangle' = \left(a - \sqrt{\frac{m\omega}{2\hbar}}x_0\right)|0\rangle'. \tag{19}$$

In other words,

$$a|0\rangle' = \sqrt{\frac{m\omega}{2\hbar}}x_0|0\rangle'. \tag{20}$$

This is an eigenequation for the annihilation operator a.

In general, the eigenstates for the annilation operator can be found as follows. Note that the annihilation operator is not Hermitian, and its eigenvalue does not have to be real. Define

$$e^{fa^\dagger}|0\rangle = \sum_{n=0}^{\infty} \frac{f^n}{n!}(a^\dagger)^n|0\rangle = \sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}}|n\rangle, \tag{21}$$

for a complex number f. If you act the annihilation operator on this state,

$$a\left(e^{fa^\dagger}|0\rangle\right) = \sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}}a|n\rangle = \sum_{n=1}^{\infty} \frac{f^n}{\sqrt{n!}}\sqrt{n}|n-1\rangle = \sum_{n=1}^{\infty} \frac{f^n}{\sqrt{(n-1)!}}|n-1\rangle. \tag{22}$$

We used the fact that $n = 0$ state does not contribute because it cannot be lowered by the annihilation operator. Changing the dummy index n to $n+1$,

$$= \sum_{n=0}^{\infty} \frac{f^{n+1}}{\sqrt{n!}}|n\rangle = f \sum_{n=0}^{\infty} \frac{f^n}{\sqrt{n!}}|n\rangle = f\left(e^{fa^\dagger}|0\rangle\right). \tag{23}$$

Therefore, this state has an eigenvalue f for the annihilation operator. We could have guessed it. The commutation relation $[a, a^\dagger] = 1$ says that roughly speaking $a = \partial/\partial a^\dagger$. Therefore, acting a just pulls out the exponent f.

We have not normalized the state yet. Working out the norm,

$$\left|e^{fa^\dagger}|0\rangle\right|^2 = \sum_{n,m}\langle n|\frac{f^{*n}}{\sqrt{n!}}\frac{f^m}{\sqrt{m!}}|m\rangle = \sum_{n} \frac{(f^*f)^n}{n!} = e^{f^*f}. \tag{24}$$

Therefore, the following state is a normalized eigenstate of the annihilation operator

$$|f\rangle = e^{-|f|^2/2}e^{fa^\dagger}|0\rangle, \qquad a|f\rangle = f|f\rangle. \tag{25}$$

This type of state is called coherent state.

Coming back to our problem, the pendulum just before the release is therefore given by the coherent state

$$|\sqrt{\frac{m\omega}{2\hbar}}x_0\rangle. \tag{26}$$

Now the interest is in its time evolution. At $t = 0$, we release the pendulum. In other words, we let the state evolve according to the original Hamiltonian without an additional force. We can address the time evolution in Heisenberg picture easier than in Schrödinger picture.

4

In Heisenberg picture, let us first study the equation of motion for the annihilation and creation operators. Because $H = \hbar\omega(a^\dagger a + \frac{1}{2})$ and $[a, a^\dagger] = 1$, we find

$$i\hbar \frac{d}{dt} a = [a, H] = \hbar\omega a. \tag{27}$$

Solving this equation is trivial,

$$a(t) = a(0)e^{-i\omega t}. \tag{28}$$

Similarly, we find

$$a^\dagger(t) = a^\dagger(0)e^{i\omega t}. \tag{29}$$

Solving the definition of the creation, annihilation operators backwards, we find the position and momentum operators

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^\dagger), \qquad p = -i\sqrt{\frac{m\hbar\omega}{2}}(a - a^\dagger). \tag{30}$$

Their time-dependence is then immediately obtained as

$$x(t) = \sqrt{\frac{\hbar}{2m\omega}}(ae^{-i\omega t} + a^\dagger e^{i\omega t}), \qquad p(t) = -i\sqrt{\frac{m\hbar\omega}{2}}(ae^{-i\omega t} - a^\dagger e^{i\omega t}). \tag{31}$$

On a coherent state, they have expectation values

$$\langle f|x(t)|f\rangle = \sqrt{\frac{\hbar}{2m\omega}}(fe^{-i\omega t} + f^*e^{i\omega t}), \tag{32}$$

$$\langle f|p(t)|f\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}(fe^{-i\omega t} - f^*e^{i\omega t}). \tag{33}$$

Note that I used $\langle f|a^\dagger = (a|f\rangle)^\dagger = (f|f\rangle)^\dagger = \langle f|f^*$. Specializing to the released pendulum, we have $f = \sqrt{\frac{m\omega}{2\hbar}}x_0$, and hence

$$\langle f|x(t)|f\rangle = x_0 \cos\omega t, \tag{34}$$
$$\langle f|p(t)|f\rangle = -m\omega x_0 \sin\omega t. \tag{35}$$

This result is the same as the classical pendulum.

Another important property of coherent states is that they have the minimum uncertainty. We can work it out easily in the following way.

$$\langle f|x|f\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle f|a + a^\dagger|f\rangle = \sqrt{\frac{\hbar}{2m\omega}}(f + f^*), \tag{36}$$

$$\langle f|p|f\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}\langle f|a - a^\dagger|f\rangle = -i\sqrt{\frac{m\hbar\omega}{2}}(f - f^*), \tag{37}$$

$$\langle f|x^2|f\rangle = \frac{\hbar}{2m\omega}\langle f|(a + a^\dagger)^2|f\rangle = \frac{\hbar}{2m\omega}(f^2 + f^{*2} + (f^*f + 1) + f^*f), \tag{38}$$

$$\langle f|p^2|f\rangle = -\frac{m\hbar\omega}{2}\langle f|(a - a^\dagger)^2|f\rangle = -\frac{m\hbar\omega}{2}(f^2 + f^{*2} - (f^*f + 1) - f^*f). \tag{39}$$

Therefore, we find

$$(\Delta x)^2 = \langle f|x^2|f\rangle - (\langle f|x|f\rangle)^2 = \frac{\hbar}{2m\omega}, \tag{40}$$

$$(\Delta p)^2 = \langle f|p^2|f\rangle - (\langle f|p|f\rangle)^2 = \frac{m\hbar\omega}{2}. \tag{41}$$

Finally, we obtain

$$\Delta x \Delta p = \frac{\hbar}{2}, \tag{42}$$

indeed the minimum uncertainty state.

To sum it up, the coherent state represents the closest approximation of a classical oscillator, with the minimum uncertainty and oscillating expectation value of the position and the momentum.

We can obtain the same result in the Schrödinger picture, which is a litte more technical than in the Heisenberg picture. The time evolution of the coherent state can be obtained as

$$\begin{aligned}
e^{-iHt/\hbar}|f\rangle &= e^{-iHt/\hbar}e^{fa^\dagger}|0\rangle e^{-|f|^2/2} \\
&= e^{-iHt/\hbar}e^{fa^\dagger}e^{iHt/\hbar}e^{-iHt/\hbar}|0\rangle e^{-|f|^2/2} \\
&= e^{fe^{-iHt/\hbar}a^\dagger e^{iHt/\hbar}}e^{-i\frac{\hbar\omega}{2}t/\hbar}|0\rangle e^{-|f|^2/2} \\
&= e^{fa^\dagger e^{-i\omega t}}|0\rangle e^{-|f|^2/2}e^{-i\omega t/2} \\
&= |fe^{-i\omega t}\rangle e^{-i\omega t/2}. \tag{43}
\end{aligned}$$

Therefore the expectation values of the postion and momentum operators are

$$
\begin{aligned}
\langle f, t | x | f, t \rangle &= \langle f e^{-i\omega t} | x | f e^{-i\omega t} \rangle \\
&= \sqrt{\frac{\hbar}{2m\omega}} (f e^{-i\omega t} + f^* e^{+i\omega t}) \\
&= x_0 \cos \omega t, \\
\langle f, t | p | f, t \rangle &= \langle f e^{-i\omega t} | p | f e^{-i\omega t} \rangle \\
&= -i \sqrt{\frac{m\hbar\omega}{2}} (f e^{-i\omega t} - f^* e^{+i\omega t}) \\
&= -m\omega x_0 \sin \omega t,
\end{aligned}
\tag{44, 45}
$$

where we used $f = \sqrt{\frac{m\omega}{2\hbar}} x_0$. The results agree with those in the Heisenberg picture Eq. (32,33).

In quantum treatment of electromagnetism, light is described by a collection of photons. For a coherent light such as laser, the electric and magnetic field behave exactly like in the classical Maxwell theory. Laser is described in terms of a coherent state.

3   Coherent State Wave Functions

Coherent state of course can be studied using the conventional wave functions. It takes a few tricks to workt it out, however.

We use the Baker–Campbell–Hausdorff formula. This is a formula important in the study of Lie algebras and Lie groups. The point is that the product of two exponentials $e^X e^Y$ can be written in terms of many commutators,

$$
\begin{aligned}
e^Z &= e^X e^Y \tag{46} \\
Z &= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}([X,[X,Y]] - [Y,[X,Y]]) \\
&\quad - \frac{1}{48}([Y,[X,[X,Y]]] + [X,[Y,[X,Y]]]) + \cdots \tag{47}
\end{aligned}
$$

See, for example, `http://en.wikipedia.org/wiki/Baker-Campbell-Hausdorff` for more details.

We use this formula for $e^{fa^\dagger}$. We take

$$X = f\sqrt{\frac{m\omega}{2\hbar}}\, x, \qquad Y = f\frac{-ip}{\sqrt{2\hbar m\omega}}. \tag{48}$$

For this purpose, we will not need any terms more than two commutators because $[X, [X, Y]] = [Y, [X, Y]] = 0$ and Eq. (47) simplifies drastically to

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]}. \tag{49}$$

Then we find

$$e^{fa^\dagger} = e^{X+Y} = e^X e^Y e^{-\frac{1}{2}[X,Y]} = e^{f\sqrt{\frac{m\omega}{2\hbar}}\, x} e^{f\frac{-ip}{\sqrt{2\hbar m\omega}}} e^{f^2/4}. \tag{50}$$

Now we are in position to work out the wave function for the coherent state.

$$
\begin{aligned}
\langle x|f\rangle &= \langle x|e^{fa^\dagger}|0\rangle e^{-|f|^2/2} \\
&= \langle x|e^{f\sqrt{\frac{m\omega}{2\hbar}}\, x} e^{f\frac{-ip}{\sqrt{2\hbar m\omega}}} e^{f^2/4}|0\rangle e^{-|f|^2/2} \\
&= e^{f\sqrt{\frac{m\omega}{2\hbar}}\, x} e^{-f\frac{\hbar}{\sqrt{2\hbar m\omega}}\frac{\partial}{\partial x}} \langle x|0\rangle e^{f^2/4} e^{-|f|^2/2} \\
&= e^{f\sqrt{\frac{m\omega}{2\hbar}}\, x} e^{-f\frac{\hbar}{\sqrt{2\hbar m\omega}}\frac{\partial}{\partial x}} \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar} e^{f^2/4} e^{-|f|^2/2} \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(f\sqrt{\frac{m\omega}{2\hbar}}\, x - \frac{m\omega}{2\hbar}\left(x - f\frac{\hbar}{\sqrt{2\hbar m\omega}}\right)^2 + \frac{f^2}{4} - \frac{|f|^2}{2}\right) \\
&= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\left(\sqrt{\frac{m\omega}{2\hbar}}\, x - f\right)^2 + \frac{1}{2}(f^2 - |f|^2)\right) \tag{51}
\end{aligned}
$$

The explicit form of the wave function allows us to calculate the shape of the probability distribution in real time. For the pulled, held, and released oscillator, the time-dependent wave function is obtained for $f = \sqrt{\frac{m\omega}{2\hbar}}x_0 e^{-i\omega t}$. Therefore,

$$\langle x|f,t\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}\left(x - x_0 e^{-i\omega t}\right)^2 + \frac{1}{2}\frac{m\omega}{2\hbar}x_0^2(e^{-2i\omega t} - 1)\right) e^{-i\omega t/2}. \tag{52}$$

We then find the probability distribution

$$
\begin{aligned}
|\psi(x,t)|^2 &= \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left(-\frac{m\omega}{\hbar}(x^2 - 2x_0 x \cos\omega t + x_0^2 \cos 2\omega t) + \frac{m\omega}{2\hbar}x_0^2(\cos 2\omega t - 1)\right) \\
&= \sqrt{\frac{m\omega}{\pi\hbar}} \exp\left(-\frac{m\omega}{\hbar}(x - x_0 \cos\omega t)^2\right). \tag{53}
\end{aligned}
$$

Therefore, it is always a Gaussian around $x_0 \cos \omega t$ which oscillators around the origin with the amplitude x_0.

4   Coherent State Representation

One important caveat about the coherent states is that they form an *over-complete* set of states. It is easy to calculate

$$\langle g | f \rangle = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \langle n | \frac{g^{*n}}{\sqrt{n!}} \frac{f^m}{\sqrt{m!}} | m \rangle = \sum_{n=0}^{\infty} \frac{(g^* f)^n}{n!} = e^{g^* f}. \tag{54}$$

Even when $g \neq f$, it does not vanish.

This is not a paradox. When we proved in class that the eigenstates of an operator with different eigenvalues are orthogonal to each other, we assumed that the operator was hermitian. The coherent states are eigenstates of the annihilation operator, which is not hermitian. Therefore, the coherent states do *not* form an orthonormal set.

Nonetheless, one can come up with the coherent state representation, taking the coherent states as the basis kets. This is because of the following completeness condition,

$$1 = \int \frac{d^2 f}{\pi} |f\rangle e^{-f^* f} \langle f|. \tag{55}$$

Here, $d^2 f = df_1 df_2$ for $f = f_1 + i f_2$ and $f_{1,2} \in \mathbb{R}$.

Let us prove the completeness relation.

$$\begin{aligned}
\int \frac{d^2 f}{\pi} |f\rangle e^{-f^* f} \langle f| &= \sum_{n,m} \int \frac{d^2 f}{\pi} e^{-f^* f} \frac{f^n}{\sqrt{n!}} |n\rangle \langle m| \frac{f^{*m}}{\sqrt{m!}} \\
&= \sum_{n,m} \int \frac{|f| d|f| d\theta}{\pi} e^{-f^* f} \frac{f^{n+m} e^{i(n-m)\theta}}{\sqrt{n!m!}} |n\rangle \langle m| \\
&= \sum_{n,m} \int \frac{|f| d|f|}{\pi} e^{-f^* f} \frac{f^{n+m} 2\pi \delta_{n,m}}{\sqrt{n!m!}} |n\rangle \langle m| \\
&= \sum_{n} \int 2|f| d|f| \frac{|f|^{2n}}{n!} e^{-|f|^2} |n\rangle \langle n| \\
&= \sum_{n} \int dt \frac{t^n}{n!} e^{-t} |n\rangle \langle n|
\end{aligned}$$

$$
\begin{aligned}
&= \sum_n \frac{1}{n!}\Gamma(n+1)|n\rangle\langle n| \\
&= \sum_n |n\rangle\langle n| = 1.
\end{aligned}
\tag{56}
$$

In the third last line, we changed the variable to $t = |f|^2$. From the completeness relation for the energy eigenstates $|n\rangle$, the last expression is indeed the unit operator.

Therefore, any state can be expressed as a linear combination of coherent states. In particular, the energy eigenstates are

$$
|n\rangle = \int \frac{d^2 f}{\pi}|f\rangle e^{-f^* f}\langle f|n\rangle = \int \frac{d^2 f}{\pi}|f\rangle e^{-f^* f} \frac{f^{*n}}{\sqrt{n!}}.
\tag{57}
$$

The coherent state representation is quite interesting because the two-dimensional integral on f can be regarded as a phase space integral. Recall the definition of the annihilation operator Eq. (2) and setting $f = a$ in this representation, we find

$$
f = \sqrt{\frac{m\omega}{2\hbar}}\left(x + \frac{ip}{m\omega}\right).
\tag{58}
$$

Therefore, we can identify

$$
\frac{d^2 f}{\pi} = \frac{1}{\pi}\sqrt{\frac{m\omega}{2\hbar}}\, dx \sqrt{\frac{m\omega}{2\hbar}}\, \frac{1}{m\omega}dp = \frac{dx\, dp}{2\pi\hbar},
\tag{59}
$$

indeed the normal phase space volume.

Let us see how one can calculate expectation values of operators using the coherent states. Note that any operator made up of x and p can be rewritten in terms of a and $a^\dagger$. Furthermore an operator can be brought to the form that all annihilation operators are moved to the left, and creation operators to the right using their commutation relations. Therefore we can cast any operators to the form $\mathcal{O} = a^n a^{\dagger m}$ without a loss of generality.* Then its expectation value can be calculated as

$$
\begin{aligned}
\langle\psi|\mathcal{O}|\psi\rangle &= \langle\psi|a^n a^{\dagger m}|\psi\rangle \\
&= \int \frac{d^2 f}{\pi}\langle\psi|a^n|f\rangle e^{-f^* f}\langle f|a^{\dagger m}|\psi\rangle
\end{aligned}
$$

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*The operators of the form $a^{\dagger m}a^n$ are said to be "normal ordered." Maybe I should call those in the order we use here "abnormally ordered."

$$= \int \frac{d^2f}{\pi} f^n f^{*m} \langle \psi|f\rangle e^{-f^*f} \langle f|\psi\rangle$$

$$= \int \frac{d^2f}{\pi} f^n f^{*m} |\langle f|\psi\rangle|^2 e^{-f^*f}. \qquad (60)$$

Therefore, the combination $|\langle f|\psi\rangle|^2 e^{-f^*f}$ can be viewed as the probability density on the phase space, where the operator $a^n a^{\dagger m}$ is simply brought to the numbers $f^n f^{*m}$.

This observation allows us to "view" a state as a probability density on the phase space. First of all, the classical motion is along a zero-thickness circle on the phase space. It is always at a point at a given moment, and the point moves along the circle as time evolves. This is shown as the first picture in Fig. 1. Note that the time corresponds to the phase, while the energy to the number.

On the other hand, the quantum mechanical energy eigenstates have the "phase space density"

$$|\langle f|n\rangle|^2 e^{-f^*f} = \left| \frac{f^{*n}}{\sqrt{n!}} \right|^2 e^{-f^*f} = \frac{|f|^{2n}}{n!} e^{-|f|^2}. \qquad (61)$$

The main support for this distribution is shown in the middle picture of Fig. 1. It basically forms a ring in the phase space with the constant energy, smeared a little bit so that the "energy" varies roughly from $n\hbar\omega$ to $(n+1)\hbar\omega$. The area is given by its uncertainty $\Delta x \Delta p = (2n+1)\hbar/2$, while the higher energy states appear as successively outward rings. The fact that it is spread out over the entire ring is a reflection of the energy-time uncertainty principle. Because we have specified energy, we don't know anything about time, and we can't say at what phase it is.

The coherent state is very close to a point on the phase space resembling the classical mechanics. The "phase space density" for the normalized state $|g\rangle e^{-g^*g/2}$ is

$$|\langle f|g\rangle e^{-g^*g/2}|^2 e^{-f^*f} = |e^{-f^*g} e^{-g^*g/2}|^2 e^{-f^*f} = e^{-|f-g|^2}. \qquad (62)$$

This is a two-dimensional Gaussian centered at $f = g$, and its main support is depicted in the right picture of Fig. 1. It has the minimum uncertainty and its area is much smaller than the energy eigenstate. The patch moves along the circle clockwise just like the classical oscillator.
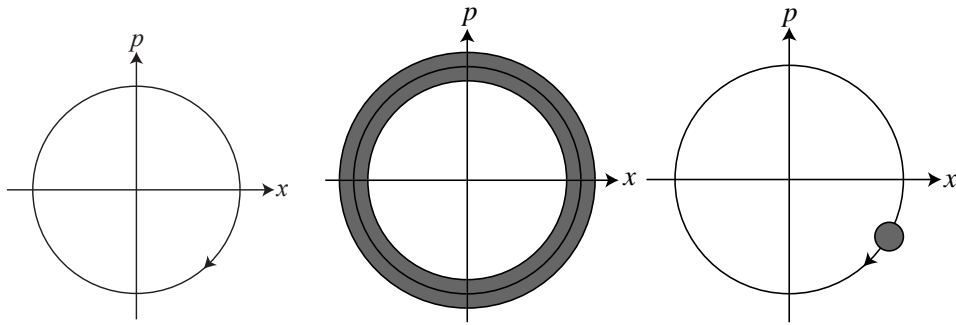
Figure 1: Classical oscillator is a point on the phase space $(x, p$ space$)$ moving along an elliptic orbit. The quantum mechanical energy eigenstate is spread out along the ellipse with no notion of motion. The uncertainty $\Delta x \Delta p$ is larger for higher levels because of a constant width around the oribit. The coherent state is a patch of the minimum uncertainty, and the whole patch moves along the classical orbit.