221A Lecture Notes on Spin

This lecture note is just for the curious. It is not a part of the standard quantum mechanics course.

1 True Origin of Spin

When we introduce spin in non-relativistic quantum mechanics, it looks ad hoc. We just say we add an additional contribution to the total angular momentum. What actually is it?

The cartoonish explanation is that particles are spinning like a top. Spin is said to be an intrinsic property of quantum mechanical particles. Spin never stops. Spin never increases. All it does is to change its orientation.

The true meaning of spin has to be discussed in the context of fully Lorentz-invariant theory. To the best of our understanding, a fully Lorentz-invariant quantum theory means relativistic quantum field theory. In such formulation, we introduce a field to every particle species, which transforms under the Lorentz transformations in a particular way. Therefore, the issue is to pick a particular representation of the Lorentz transformations. Once that is done, it specifies spin. After quantizing the field, you find that the field operator can create or annihilate a particle of definite spin. We will see this from Dirac equation and Maxwell equation in 221B.

Because the spin is already a part of the field, you can't say that you've added spin to the total angular momentum. It is there from the first place. In popular science magazines, you may say that spin is like the particle spinning around its axis just like the Earth does. Well, it is indeed like that. But you can't push this picture too far. Lorentz tried to buid a theory of electron assuming that it is a sphere of finite radius. Then you may be able to understand its mass, or its rest energy (remember $E = mc^2$!), as an electrostatic energy of concentrated negative charge. Then the radius must be roughly $r_e \sim e^2/(m_e c^2) \sim 10^{-13}$ cm. Imagine further that this sphere is spinning. To obtain the spin of $\hbar/2$, a dimensional analysis suggests $m_e r_e v \sim \hbar$, where v is the speed of rotation at the surface of the sphere. Then we find $v \sim \hbar/(m_e r_e) \sim \hbar c^2/e^2 \sim 137c \gg c$! Well, it just doesn't work.

Of course spins can indeed come from certain internal structure. The spins of atoms are the sum of nucleus and elelectron spins together with the orbital angular momenta. The spins of nuclei are the sum of proton and neutron spins together with the orbital angular momenta. The spin of the proton is the sum of quark spins (with no apparent contribution from the orbital angular momentum). But you still ask: where do the quark spins come from? The best answer is that when you get a quark by quantizing a quark field, the field already carries spin. It is just a property already there when you introduce a field.

How do we choose spin when you introduce a field, then? A consistent (*i.e.*, renormalizable) quantum field theory can include only spin 0, 1/2, and 1. Renormalizable interactions are only those interactions that can appear without extra suppressions of $G_N E^2/\hbar c^5 = (E/10^{19}\text{ GeV})^2 \ll 1$. Therefore, there is a reason why we see only particles of spin 1/2 and 1. Well, what about spin 0? We have not seen any fundamental particle of spin 0 yet. We are looking for one: the Higgs boson, which is expected to permeate our entire Universe, dragging the foot of all quarks, leptons, W and Z bosons, making it impossible for them to reach the speed of light. It is expected to be found this decade thanks to higher-energy accelerators being built. Once it is found, it exhausts all theoretical possibilities of spin 0, 1/2, and 1.

What about gravity? If you manage to quantize gravity consistently, you will find a spin 2 particle: graviton. The trouble is, to this date, nobody managed to do this. It doesn't give you a renormalizable theory, or in other words, you get all kinds of seemingly meaningless infinities you don't know how to get rid of. But we know one thing: even at the classical level, fields with spin higher than 2 cannot have interactions consistently. You find there are way too many components of the field you can't get rid of, and the theory becomes unphysical (*i.e.*, sick).

The string theory is arguably a consistent theory of quantum gravity. There are no elementary particles; what we think are particles are actually string curled up to a tiny size. It predicts that there are states with arbitrarily higher spins, but their masses are all up at the Planck scale, 10^{19} GeV/c^2. High spins come out because the string can spin around. All our lowly existence must be made up of lower spin states, where only particles of spin above 1 are the graviton (spin 2) and its superpartner gravitino (spin 3/2). We are not supposed to see higher spin particles unless somebody figures how to produce these "particles" as heavy as bacteria.

The rest of this note is devoted to (somewhat academic) discussions on how you can view spin as an additional degrees of freedom for a particle. They may give you more insight into the nature of spin and angular momentum in general.

2 Classical Lagrangian for Spin

It is possible to obtain any spin from a Lagrangian. Spin always has a fixed size and keeps spinning eternally. If you imagine spin as a little arrow with fixed length sticking out from the particle, the dynamical degrees of freedom are at which direction it is pointing out. Therefore, it is reasonable to expect that dynamical variables are polar and azimuthal angles θ and ϕ. We imagine that every particle comes together with this "arrow" in addition to its position and momentum. The rest is the actual construction.

You take the *phase space* to be a surface of a sphere (S^2) parameterized by the polar angle θ and the azimuth ϕ, and take the action

$$S = J \int \cos\theta\dot{\phi}dt. \tag{1}$$

Note that the sphere is not the coordinate space but the phase space. Correspondingly, the Lagrangian has only one time derivative, not two. This term corresponds to $\int p_i \dot{q}_i dt$ in more conventional systems. You can regard ϕ to be the "canonical coordinate", while $J\cos\theta$ to be the "canonical momentum." The important difference from other systems is that this phase space has a finite volume. Because, semi-classically, the number of states is given by the phase space volume in the unit of $2\pi\hbar$ per degree of freedom, a finite-volume phase space implies a finite dimensional Hilbert space. On the other hand, S^2 is invariant under three-dimensional rotation, and the symmetry guarantees that there arises angular momentum operators with the correct commutation relations.*

Once we add this term to the Lagrangian of a point particle, it can describe a particle with spin J. I emphasize that θ and ϕ form an additional phase space, in addition to the particle's ordinary phase space $(\vec{x}, \vec{p})$.

The Poisson bracket is

$$\{\phi, J\cos\theta\} = 1, \tag{2}$$

or in other words, it is defined by

$$\{A, B\} = -\frac{1}{J\sin\theta}\left(\frac{\partial A}{\partial\phi}\frac{\partial B}{\partial\theta} - \frac{\partial A}{\partial\theta}\frac{\partial B}{\partial\phi}\right). \tag{3}$$

*If you are familiar with differential geometry, this is where this expression for the action comes from. You first write the volume form $\omega = J\sin\theta d\phi \wedge d\theta$. This is clearly invariant under rotations. Because ω is closed, you can write it locally as an exact form $\omega = d\chi = d(J\cos\theta d\phi)$. Then the action is $S = \int \chi$. This construction guarantees that it is rotationally invariant up to a surface term.

From this definition, we can work out the Poisson brackets among

$$
\begin{aligned}
J_x &= J \sin\theta \cos\phi, & (4)\\
J_y &= J \sin\theta \sin\phi, & (5)\\
J_z &= J \cos\theta. & (6)
\end{aligned}
$$

We find

$$
\begin{aligned}
\{J_x, J_y\} &= -\frac{1}{J\sin\theta}((-J\sin\theta\sin\phi)(J\cos\theta\sin\phi) \\
&\quad -(J\cos\theta\cos\phi)(J\sin\theta\cos\phi)) = J\cos\theta = J_z, & (7)\\
\{J_y, J_z\} &= -\frac{1}{J\sin\theta}((J\sin\theta\cos\phi)(-J\sin\theta)) = J\sin\theta\cos\phi = J_x, & (8)\\
\{J_z, J_x\} &= -\frac{1}{J\sin\theta}(-(-J\sin\theta)(-J\sin\theta\sin\phi)) = J\sin\theta\sin\phi = J_y. & (9)
\end{aligned}
$$

Or with tensor notation,

$$\{J_k, J_l\} = \epsilon_{klm} J_m. \tag{10}$$

We can also add a Hamiltonian to it. For instance, the magnetic moment interaction in a constant magnetic field is

$$H = -\vec{\mu}\cdot\vec{B} = -\mu(B_x \sin\theta\cos\phi + B_y\sin\theta\sin\phi + B_z\cos\theta) = -\frac{\mu}{J}\vec{B}\cdot\vec{J}. \tag{11}$$

The corresponding action is

$$S = \int (J\cos\theta\dot{\phi} + \frac{\mu}{J}\vec{B}\cdot\vec{J})dt. \tag{12}$$

The Hamilton equation of motion is

$$\frac{dJ_k}{dt} = \{J_k, H\} = -\frac{\mu}{J}\{J_k, B_l J_l\} = -\frac{\mu}{J} B_l \epsilon_{klm} J_m, \tag{13}$$

or with vector notation,

$$\frac{d\vec{J}}{dt} = -\frac{\mu}{J}\vec{B}\times\vec{J}. \tag{14}$$

This is indeed the equation for spin precession.

3 Quantum Theory of Spin

In this section, we quantize the classical action for spin introduced in the previous section. This type of construction of quantum spins had actually been used in Haldane's theory of anti-ferromagnetism in one-dimensional spin chain.

3.1 Allowed Values for J

We first write $J = j\hbar$ for later convenience,

$$S = j\hbar \int \cos\theta \dot{\phi} dt. \tag{15}$$

It is easy to see that $2j$ must be an integer, following the same type of arguments as the charged quantization in the presence of a monopole. Consider a closed path C on the phase space, and its action

$$S = j\hbar \oint_C \cos\theta d\phi. \tag{16}$$

It can be rewritten as a surface integral using Stokes' theorem over a surface M ($\partial M = C$),

$$S = j\hbar \int_M \sin\theta d\phi d\theta. \tag{17}$$

But there are two choices for M on each side of the loop. The difference between two choices is nothing but the integral over the entire sphere,

$$\Delta S = j\hbar \int_{S^2} \sin\theta d\theta d\phi = j\hbar 4\pi. \tag{18}$$

We want $e^{iS/\hbar}$ to be single-valued, and hence $e^{i\Delta S/\hbar} = 1$. It means $4\pi j = 2\pi N$ ($N \in \mathbb{Z}$), or $2j = N$.

3.2 Angular Momentum Operators

When quantizing it, we have to pay a careful attention to the ordering of the operators. The definition of $J_z = j\hbar \cos\theta$ remains the same as in the classical case. The problem is that $\sin\theta$ and ϕ do not commute in the definition of

J_x, J_y. You have to go through a bit of trial and error to get a consistent definition. In the end, here is what you need:

$$J_+ = \sqrt{j\hbar(1+\cos\theta)}\ e^{i\phi}\sqrt{j\hbar(1-\cos\theta)}, \tag{19}$$
$$J_- = \sqrt{j\hbar(1-\cos\theta)}\ e^{-i\phi}\sqrt{j\hbar(1+\cos\theta)}. \tag{20}$$

To the extent you ignore the ordering, this is consistent with the classical expressions $J_\pm = J\sin\theta e^{\pm i\phi}$. We will see why this works by checking the commutation relations. The canonical commutation relation is

$$[\phi, j\hbar\cos\theta] = i\hbar. \tag{21}$$

From this, we can say that

$$j\hbar\cos\theta \simeq \frac{\hbar}{i}\frac{\partial}{\partial\phi}, \tag{22}$$

where the reason for $\simeq$ but not $=$ becomes clear later. Then clearly,

$$[J_z, e^{\pm i\phi}] = \pm\hbar e^{\pm i\phi}. \tag{23}$$

Given this, it is easy to verify $[J_z, J_\pm] = \pm\hbar J_\pm$. The tricky one is $[J_+, J_-]$. Paying careful attention to the ordering of operators,

$$\begin{aligned}
&[J_+, J_-] \\
&= \sqrt{j\hbar(1+\cos\theta)}\ e^{i\phi}\sqrt{j\hbar(1-\cos\theta)}\sqrt{j\hbar(1-\cos\theta)}\ e^{-i\phi}\sqrt{j\hbar(1+\cos\theta)} \\
&\quad -\sqrt{j\hbar(1-\cos\theta)}\ e^{-i\phi}\sqrt{j\hbar(1+\cos\theta)}\sqrt{j\hbar(1+\cos\theta)}\ e^{i\phi}\sqrt{j\hbar(1-\cos\theta)} \\
&= \sqrt{j\hbar(1+\cos\theta)}\ e^{i\phi}j\hbar(1-\cos\theta)e^{-i\phi}\sqrt{j\hbar(1+\cos\theta)} \\
&\quad -\sqrt{j\hbar(1-\cos\theta)}\ e^{-i\phi}j\hbar(1+\cos\theta)e^{i\phi}\sqrt{j\hbar(1-\cos\theta)} \\
&= \sqrt{j\hbar(1+\cos\theta)}\ e^{i\phi}e^{-i\phi}(j\hbar(1-\cos\theta)+\hbar)\sqrt{j\hbar(1+\cos\theta)} \\
&\quad -\sqrt{j\hbar(1-\cos\theta)}\ e^{-i\phi}e^{i\phi}(j\hbar(1+\cos\theta)+\hbar)\sqrt{j\hbar(1-\cos\theta)} \\
&= \sqrt{j\hbar(1+\cos\theta)}\ (j\hbar(1-\cos\theta)+\hbar)\sqrt{j\hbar(1+\cos\theta)} \\
&\quad -\sqrt{j\hbar(1-\cos\theta)}\ (j\hbar(1+\cos\theta)+\hbar)\sqrt{j\hbar(1-\cos\theta)} \\
&= 2j\hbar^2\cos\theta = 2\hbar J_z.
\end{aligned} \tag{24}$$

This is exactly what we need.

The above calculation also allows us to obtain $\vec{J}^2$,

$$\begin{aligned}
\vec{J}^2 &= J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) \\
&= (j\hbar\cos\theta)^2 + \frac{1}{2}\left[\sqrt{j\hbar(1+\cos\theta)}\ (j\hbar(1-\cos\theta)+\hbar)\sqrt{j\hbar(1+\cos\theta)}\right. \\
&\quad \left. -\sqrt{j\hbar(1-\cos\theta)}\ (j\hbar(1+\cos\theta)+\hbar)\sqrt{j\hbar(1-\cos\theta)}\right] \\
&= j(j+1)\hbar^2. \qquad (25)
\end{aligned}$$

3.3 Wave Functions

We have to avoid any sorts of singularities, making sure that $p_i dq_i$ is well-defined. It turns out, however, $\cos\theta d\phi$ is not well-defined where $\cos\theta = \pm 1$, or at poles. There, anything proportional to $d\phi$ must vanish because ϕ loses its meaning. To make sure that is the case, we have to add a total derivative $\pm j\hbar\dot{\phi}$ to the Lagrangian, but it works only for one of the poles, not both. Therefore, we use $j\hbar(-1+\cos\theta)$ in the northern hemisphere, and use $j\hbar(1+\cos\theta)$ in the southern hemisphere. We connect the two at the equator by adding a total derivative to the Lagrangian $\pm 2j\hbar\dot{\phi}$, which corresponds to a gauge transformation by $e^{\pm i2j\phi}$ on wave functions.

We start with the southern hemisphere (rejoice, Aussies!) There, we start with the action

$$S = j\hbar \int (1+\cos\theta)\dot{\phi}dt. \qquad (26)$$

Therefore, the canonical commutation relation is

$$[\phi, j\hbar(1+\cos\theta)] = i\hbar. \qquad (27)$$

In the coordinate representation where ϕ is diagonal, we have

$$j\hbar(1+\cos\theta) = \frac{\hbar}{i}\frac{\partial}{\partial\phi}. \qquad (28)$$

Because $J_z = j\hbar\cos\theta$, we find

$$J_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} - j\hbar. \qquad (29)$$

Clearly the eigenstates of J_z with the eigenvalue $m\hbar$ are given by

$$\psi_m(\phi) = \frac{1}{\sqrt{2\pi}} e^{i(j+m)\phi}. \qquad (30)$$

Note that this form of $\psi_m(\phi)$ does satisfy the periodic boundary condition $\psi_m(\phi+2\pi) = \psi_m(\phi)$ even for half-integer j because of the combination $j+m$. The raising and lowering operators are then given by

$$J_+ = \sqrt{\frac{\hbar}{i}\frac{\partial}{\partial\phi}}\; e^{i\phi}\sqrt{2j\hbar - \frac{\hbar}{i}\frac{\partial}{\partial\phi}}, \tag{31}$$

$$J_- = \sqrt{2j\hbar - \frac{\hbar}{i}\frac{\partial}{\partial\phi}}\; e^{-i\phi}\sqrt{\frac{\hbar}{i}\frac{\partial}{\partial\phi}}. \tag{32}$$

It is straightforward to verify that

$$J_+\frac{1}{\sqrt{2\pi}}e^{i(j+m)\phi} = \sqrt{j(j+1)-m(m+1)}\;\frac{1}{\sqrt{2\pi}}e^{i(j+m+1)\phi}, \tag{33}$$

$$J_-\frac{1}{\sqrt{2\pi}}e^{i(j+m)\phi} = \sqrt{j(j+1)-m(m-1)}\;\frac{1}{\sqrt{2\pi}}e^{i(j+m-1)\phi}. \tag{34}$$

In the northern hemisphere, we start with the action

$$S = j\hbar\int(-1+\cos\theta)\dot{\phi}dt. \tag{35}$$

Therefore, the canonical commutation relation is

$$[\phi, j\hbar(-1+\cos\theta)] = i\hbar. \tag{36}$$

In the coordinate representation where ϕ is diagonal, we have

$$j\hbar(-1+\cos\theta) = \frac{\hbar}{i}\frac{\partial}{\partial\phi}. \tag{37}$$

Because $J_z = j\hbar\cos\theta$, we find

$$J_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi} + j\hbar. \tag{38}$$

Clearly the eigenstates of J_z with the eigenvalue $m\hbar$ are given by

$$\psi_m(\phi) = \frac{1}{\sqrt{2\pi}}e^{i(-j+m)\phi}. \tag{39}$$

This wave function is related to that in the southern hemispher by a "gauge transformation" $e^{-2ij\phi}$, as expected from the difference in the total derivative term $\Delta L = -2j\hbar\dot{\phi}$. The raising and lowering operators are then

$$J_+ = \sqrt{2j\hbar + \frac{\hbar}{i}\frac{\partial}{\partial\phi}}\, e^{i\phi}\sqrt{-\frac{\hbar}{i}\frac{\partial}{\partial\phi}}, \tag{40}$$

$$J_- = \sqrt{-\frac{\hbar}{i}\frac{\partial}{\partial\phi}}\, e^{-i\phi}\sqrt{2j\hbar + \frac{\hbar}{i}\frac{\partial}{\partial\phi}}. \tag{41}$$

It is straightforward to verify that

$$J_+ \frac{1}{\sqrt{2\pi}} e^{i(j+m)\phi} = \sqrt{j(j+1) - m(m+1)}\, \frac{1}{\sqrt{2\pi}} e^{i(j+m+1)\phi}, \tag{42}$$

$$J_- \frac{1}{\sqrt{2\pi}} e^{i(j+m)\phi} = \sqrt{j(j+1) - m(m-1)}\, \frac{1}{\sqrt{2\pi}} e^{i(j+m-1)\phi}. \tag{43}$$

3.4 Working Backwards

It may be instructive to see how the action we used can be justified back from the representation of spin we know.

Let us first study the state

$$|z\rangle \equiv e^{zJ_-/\hbar}|j,j\rangle. \tag{44}$$

By Taylor expanding the operator, we find

$$\begin{aligned} |z\rangle &= \sum_{n=0}^{\infty} \frac{z^n}{n!\hbar^n}(J_-)^n|j,j\rangle = \sum_{n=0}^{2j} \frac{z^n}{n!}\sqrt{\frac{(2j)!n!}{(2j-n)!}}\,|j,j-n\rangle \\ &= \sum_{m=-j}^{j} z^{j-m}\sqrt{\frac{(2j)!}{(j+m)!(j-m)!}}\,|j,m\rangle. \end{aligned} \tag{45}$$

They have inner products

$$\langle z_1|z_2\rangle = (1 + \bar{z}_1 z_2)^{2j}. \tag{46}$$

The important property of this state is that the spin is oriented along a particular direction. This can be seen first by studying the angular momentum operators. Using Hausdorff formula

$$e^A B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[A,[A,\cdots[A}_{n},B]\cdots]] \tag{47}$$

we obtain

$$
\begin{aligned}
e^{-zJ_-/\hbar} J_z e^{zJ_-/\hbar} &= J_z + [J_z, \frac{z}{\hbar} J_-] = J_z - zJ_-, && (48)\\
e^{-zJ_-/\hbar} J_- e^{zJ_-/\hbar} &= J_0, && (49)\\
e^{-zJ_-/\hbar} J_+ e^{zJ_-/\hbar} &= J_+ + [J_+, \frac{z}{\hbar} J_-] + \frac{1}{2!}[[J_+, \frac{z}{\hbar} J_-], \frac{z}{\hbar} J_-] = J_+ + 2zJ_z - z^2 J_-. && (50)
\end{aligned}
$$

This allows us to find

$$
\begin{aligned}
(J_z + zJ_-)|z\rangle &= j\hbar|z\rangle, && (51)\\
(J_+ - zJ_z)|z\rangle &= j\hbar z|z\rangle. && (52)
\end{aligned}
$$

Then

$$[(J_z + zJ_-) + \bar{z}(J_+ - zJ_z)]|z\rangle = j\hbar(1 + z\bar{z})|z\rangle. \qquad (53)$$

Writing $z = re^{i\phi}$ and recalling $J_\pm = (J_x \pm iJ_y)$, we obtain

$$\left[\frac{2r}{1+r^2}(J_x \cos\phi + J_y \sin\phi) + \frac{1-r^2}{1+r^2} J_z\right]|z\rangle = j\hbar|z\rangle. \qquad (54)$$

We can further rewrite it with

$$\sin\theta = \frac{2r}{1+r^2}, \qquad \cos\theta = \frac{1-r^2}{1+r^2} \qquad (55)$$

as

$$[J_x \sin\theta\cos\phi + J_y \sin\theta\sin\phi + J_z\cos\theta]\,|z\rangle = j\hbar|z\rangle. \qquad (56)$$

In other words, the state $|z\rangle$ is the eigenstate of $\vec{J}\cdot\vec{n} = J_x \sin\theta\cos\phi + J_y \sin\theta\sin\phi + J_z\cos\theta$ along the orientation defined by the polar angle θ and the azimuth ϕ.† As z sweeps the complex plane, the spin can orient at any directions.

We can decompose unity with these states

$$1 = \frac{2j+1}{\pi}\int \frac{d^2 z}{(1+z\bar{z})^{2j+2}}|z\rangle\langle z|. \qquad (57)$$

† z is the projective coordinate of a sphere, *i.e.*, the projection of sphere on a plane attached at the north pole using rays shone from the south pole.

This can be shown as follows. Writing $z = re^{i\phi}$, $d^2z \equiv rdrd\phi$,

$$
\begin{aligned}
&\frac{2j+1}{\pi}\int \frac{d^2z}{(1+z\bar{z})^{2j+2}}|z\rangle\langle z| \\
&= \frac{2j+1}{\pi}\int \frac{d^2z}{(1+z\bar{z})^{2j+2}} \\
&\quad \sum_{m,m'} z^{j-m}\sqrt{\frac{(2j)!}{(j+m)!(j-m)!}}\ |j,m\rangle\langle j,m'|\sqrt{\frac{(2j)!}{(j+m')!(j-m')!}}\ \bar{z}^{j-m'} \\
&= \frac{2j+1}{\pi}\int \frac{rdrd\phi}{(1+r^2)^{2j+2}}\sum_{m,m'}(r)^{2j-m-m'}e^{i(m'-m)\phi} \\
&\quad \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}}\sqrt{\frac{(2j)!}{(j+m')!(j-m')!}}\ |j,m\rangle\langle j,m'| \\
&= \frac{2j+1}{\pi}\int_0^\infty \frac{2\pi rdr}{(1+r^2)^{2j+2}}\sum_m (r^2)^{j-m}\frac{(2j)!}{(j+m)!(j-m)!}|j,m\rangle\langle j,m| \\
&= (2j+1)\sum_m \frac{(2j)!}{(j+m)!(j-m)!}|j,m\rangle\langle j,m|\int_0^\infty \frac{dr^2}{(1+r^2)^{2j+2}}(r^2)^{j-m} \\
&= (2j+1)\sum_m \frac{(2j)!}{(j+m)!(j-m)!}|j,m\rangle\langle j,m|\int_0^\infty \frac{t^{j-m}dt}{(1+t)^{2j+2}} \\
&= \sum_m \frac{(2j+1)!}{(j+m)!(j-m)!}B(j-m+1,j+m+1)|j,m\rangle\langle j,m| \\
&= \sum_m |j,m\rangle\langle j,m| \\
&= 1. \qquad (58)
\end{aligned}
$$

The repeated insertion of the decomposition of unity leads to a path integral. We ignore the overall factor $((2j+1)/\pi)^N$:

$$
\begin{aligned}
\langle z_f|z_i\rangle &= \prod_k \int \frac{d^2z_i}{(1+\bar{z}_i z_i)^{2j+2}}\langle z_f|z_N\rangle\langle z_N|z_{N-1}\rangle\cdots\langle z_2|z_1\rangle\langle z_1|z_i\rangle \\
&= \prod_k \int \frac{d^2z_i}{(1+\bar{z}_i z_i)^{2j+2}}(1+\bar{z}_f z_N)^{2j}(1+\bar{z}_N z_{N-1})^{2j}\cdots(1+\bar{z}_1 z_i)^{2j} \\
&= \prod_k \int \frac{d^2z_i}{(1+\bar{z}_i z_i)^2}\exp 2j\left(\ln(1+\bar{z}_f z_N) - \ln(1+\bar{z}_N z_N)\right. \\
&\quad \left. + \ln(1+\bar{z}_N z_{N-1}) - \ln(1+\bar{z}_{N-1}z_{N-1})\cdots - \ln(1+\bar{z}_1 z_1) + \ln(1+\bar{z}_1 z_i)\right). \qquad (59)
\end{aligned}
$$

The limit of infinite time slices $N \to \infty$ makes the exponent

$$2j \int dt \frac{\dot{\bar{z}} z}{1 + \bar{z} z}. \tag{60}$$

This must be $iS/\hbar$ of the classical action.

Now we rewrite this path integral in terms of θ and ϕ. Using the definition of θ from $\cos\theta = \frac{1-r^2}{1+r^2}$,

$$d\cos\theta = d\frac{1-r^2}{1+r^2} = -\frac{4r dr}{(1+r^2)^2}. \tag{61}$$

Therefore,

$$\frac{d^2 z}{(1+\bar{z}z)^2} = \frac{r dr d\phi}{(1+r^2)^2} = \frac{1}{4} d\cos\theta d\phi. \tag{62}$$

Therefore the path integral is just a successive integration over the surface of the sphere. It is also useful to know $r = \tan\frac{\theta}{2}$. The action is

$$\begin{aligned} \frac{i}{\hbar} S &= 2j \int dt \frac{\dot{\bar{z}} z}{1 + \bar{z} z} \\ &= 2j \int dt \cos^2\frac{\theta}{2} \left(\frac{\dot{\theta}}{2\cos^2\frac{\theta}{2}} e^{-i\phi} - i \tan\frac{\theta}{2} \dot{\phi} e^{-i\phi} \right) \tan\frac{\theta}{2} e^{i\phi} \\ &= 2j \int dt \left(\frac{1}{2} \tan\frac{\theta}{2} \dot{\theta} - i \sin^2\frac{\theta}{2} \dot{\phi} \right). \end{aligned} \tag{63}$$

The first term is a total derivative $\frac{d}{dt} \ln\cos\frac{\theta}{2}$ and can be dropped. The second term can be simplified using $\sin^2\frac{\theta}{2} = (1-\cos\theta)/2$ and dropping a total derivative,

$$\frac{i}{\hbar} S = ij \int dt \cos\theta \dot{\phi}. \tag{64}$$

This is precisely the action we had used in previous sections.

3.5 Geometric Quantization

This section is for mathematically inclined. Quantization of systems with compact phase space can be done consistently with a formalism called "geometric quantization." In this formalism, the action is an integral of a one-form $\chi = p_i dq_i$ on the phase space, which gives the symplectic form of the phase space as $\omega = d\chi$. The method is particularly clear when the phase space admits a Kähler structure. Then the sympletic form is nothing but the Kähler form. You then construct a holomorphic line bundle on the phase space whose first Chern class is given by the Kähler form. This of course requires that the Kähler form belongs to the second cohomology of the symplectic manifold with integer coefficient, *i.e.*, the phase space volume is quantized. Obtain all holomorphic sections of the line bundle. They form the finite-dimensional Hilbert space. In our case, $S^2 \simeq \mathbb{C}P^1$, which admits a Kähler structure. Using a complex coordinate z, the Kähler potential is $K = 2j \ln(1 + \bar{z}z)$. The sympletic two-form is nothing but the Käher two-form $\omega = i\partial\bar{\partial}K = i\frac{2j dz \wedge d\bar{z}}{(1+\bar{z}z)^2}$. The gauge connection is obtained from the requirement that $\omega = \bar{\partial}A$ with $\bar{A} = 0$, and hence

$$A = -2j \frac{i\bar{z}dz}{1 + \bar{z}z}. \tag{65}$$

Between two patches, the coordinate transformation $z \to -1/z$ gives

$$A = 2j \frac{idz}{1 + \bar{z}z} \frac{1}{z}, \tag{66}$$

clearly singular at $z = 0$. Therefore a transition function of z^{-2j} is needed, to obtain

$$A = 2j \frac{idz}{1 + \bar{z}z} \frac{1}{z} - iz^{2j}\partial z^{-2j} = -2j \frac{i\bar{z}dz}{1 + \bar{z}z}. \tag{67}$$

A holomorphic section regular at $z = 0$ are given by positive powers in z,

$$1, z, z^2, \cdots, z^N, \cdots \tag{68}$$

but at the infinity, we multiply them with the transition function and find

$$z^{-2j}, z^{-2j+1}, z^{-2j+2}, \cdots, z^{-2j+N}, \cdots \tag{69}$$

They are regular at $z = \infty$ only for $N \leq 2j$. Therefore, we obtain $2j + 1$ dimensional vector space of holomorphic sections,

$$1, z, z^2, \cdots, z^{2j}. \tag{70}$$

The operators are given by

$$J_z = z\partial - j, \qquad J_+ = -z^2\partial + 2jz, \qquad J_- = \partial. \tag{71}$$

The state $|j,m\rangle$ is represented by z^{j+m}.

The inner product of wave functions is defined obviously by an integral with the volume factor $\frac{d^2z}{(1+\bar{z}z)^2}$. However, to ensure that the integrand is invariant under the coordinate transformation $z \to -1/z$ together with the transition function z^{2j}, we need another factor $\frac{1}{(1+\bar{z}z)^{2j}}$, so that

$$\frac{\psi^*(\bar{z})\psi(z)}{(1+\bar{z}z)^{2j}} \to \frac{\psi^*(-1/\bar{z})\psi(-1/z)}{(1+1/(\bar{z}z))^{2j}} = \frac{\bar{z}^{2j}\psi^*(-1/\bar{z})z^{2j}\psi(-1/z)}{(1+\bar{z}z)^{2j}}. \tag{72}$$

Because

$$\int \frac{d^2z}{(1+\bar{z}z)^2}\frac{\bar{z}^k z^{k'}}{(1+\bar{z}z)^{2j}} = \int \frac{rdrd\phi}{(1+r^2)^{2j+2}} r^{k+k'} e^{i(k'-k)\phi} = \pi\frac{k!(2j-k)!}{(2j+1)!}\delta_{k,k'}. \tag{73}$$

Therefore the correctly normalized states are given by

$$\psi_m(z) = \sqrt{\frac{2j+1}{\pi}}\sqrt{\frac{(2j)!}{k!(2j-k)!}}z^k = \sqrt{\frac{2j+1}{\pi}}\sqrt{\frac{(2j)!}{(j+m)!(j-m)!}}\ z^{j+m}. \tag{74}$$

Relationship to the backward construction in the previous section is obvious.

4 Other Constructions of Spin

There had been many attempts to simplify our life dealing with spins (or in general, angular momenta). Here are some.

Sakurai introduces Schwinger's method to deal with angular momentum. Is it basically that you have two harmonic oscillators

$$H = \hbar\omega(a_1^\dagger a_1 + a_2^\dagger a_2 + 1) \tag{75}$$

and assigning spin up to $a_1^\dagger$, spin down to $a_2^\dagger$, *i.e.*, spin 1/2 to this doublet. Using spin 1/2 many many times can give you any spin you'd like. This way, you can construct Hilbert space of angular momentum from simple harmonic oscillators.

If you already know what j you are interested in, you can live with only one harmonic oscillator. You start with the standard vacuum, $a|0\rangle = 0$ identified as $|j,-j\rangle = |0\rangle$. Now you define

$$J_z = \hbar(a^\dagger a - j), \tag{76}$$

$$
\begin{aligned}
J_+ &= \hbar a^\dagger \sqrt{2j - a^\dagger a}, & (77)\\
J_- &= \hbar \sqrt{2j - a^\dagger a}\; a. & (78)
\end{aligned}
$$

It looks odd, but it works. Checking commutation relations $[J_z, J_\pm] = \pm\hbar J_\pm$ is easy. The tricky one is

$$
\begin{aligned}
[J_+, J_-] &= \hbar^2 a^\dagger \sqrt{2j - a^\dagger a}\sqrt{2j - a^\dagger a}\; a - \hbar^2 \sqrt{2j - a^\dagger a}\; a a^\dagger \sqrt{2j - a^\dagger a} \\
&= \hbar^2 a^\dagger (2j - a^\dagger a) a - \hbar^2 \sqrt{2j - a^\dagger a}(a^\dagger a + 1)\sqrt{2j - a^\dagger a} \\
&= \hbar^2 (2j - a^\dagger a + 1) a^\dagger a - \hbar^2 (2j - a^\dagger a)(a^\dagger a + 1) \\
&= 2\hbar^2 (a^\dagger a - j) = 2\hbar J_z. \qquad (79)
\end{aligned}
$$

Because the commutation relations come out correctly, it is guaranteed that their matrix representations are also correct, using $|j, m\rangle = |j + m\rangle$ in the harmonic oscillator languate. The factor $\sqrt{2j - a^\dagger a}$ basically truncates the Hilbert space at $a^\dagger a = 2j$ by a brute force. But this formalism does not come out from a Lagrangian or Hamiltonian formulation; it is also odd that there is no way to flip J_z, $m \leftrightarrow -m$.