221B Lecture Notes
Scattering Theory II

1 Born Approximation

Lippmann–Schwinger equation

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0 + i\epsilon} V |\psi\rangle, \tag{1}$$

is an exact equation for the scattering problem, but it still is an equation to be solved because the state vector $|\psi\rangle$ appears on both sides of the equation. In the coordinate space, as we derived in Scattering Theory I, it becomes

$$\psi(\vec{x}) \simeq \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{k}\cdot\vec{x}} - \frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d\vec{x'} e^{-i\vec{k'}\cdot\vec{x'}} V(\vec{x'})\psi(\vec{x'}), \tag{2}$$

far away from the scatterer where $r = |\vec{x}|$ and $\vec{k'} = \vec{k}\frac{\vec{x}}{r}$ is the wave-vector of the scattered wave. Note that $|\vec{k'}| = \vec{k}$. It is an integral equation for the unknown function $\psi(\vec{x})$.

One way to solve the Lippmann–Schwinger equation Eq. (1) is by perturbation theory, *i.e.*, a power series expansion in the potential V. Note that, in the absence of the potential, $|\psi\rangle = |\phi\rangle$, or in other words, $|\psi\rangle = |\phi\rangle + O(V)$. Therefore the lowest (1st) order approximation in V is write

$$|\psi\rangle = |\phi\rangle + \frac{1}{E - H_0 + i\epsilon} V |\phi\rangle + O(V^2), \tag{3}$$

and neglect $O(V^2)$ correction. This is called *Born approximation*,[1] or more correctly, 1st Born approximation. Obviously, this approximation is good only when the scattering is weak.

In the coordinate space, we again replace ψ by ϕ in the r.h.s. of Eq. (2), and find

$$\begin{aligned}\psi(\vec{x}) &\simeq \frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{k}\cdot\vec{x}} - \frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d\vec{x'} e^{-i\vec{k'}\cdot\vec{x'}} V(\vec{x'})\frac{1}{(2\pi\hbar)^{3/2}} e^{i\vec{k}\cdot\vec{x'}} \\ &= \frac{1}{(2\pi\hbar)^{3/2}}\left[e^{i\vec{k}\cdot\vec{x}} - \frac{2m}{\hbar^2}\frac{e^{ikr}}{4\pi r}\int d\vec{x'} V(\vec{x'}) e^{i\vec{q}\cdot\vec{x'}}\right],\end{aligned} \tag{4}$$

[1] Did you know that Max Born is the grandfather of Olivia Newton-John? See, *e.g.*, `http://mooni.fccj.org/~ethall/trivia/trivia.htm`.

where $\vec{q} = \vec{k} - \vec{k'}$ is the *momentum transfer* in the scattering process.

The expression Eq. (4) is very interesting. It shows that the scattering amplitude is the Fourier transform of the potential,

$$f^{(1)}(\vec{k'}, \vec{k}) = -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d\vec{x} V(\vec{x}) e^{i\vec{q}\cdot\vec{x}}, \tag{5}$$

up to a numerical factor of $-(1/4\pi)(2m/\hbar^2)$. The superscript shows that this is a result valid at the first order in V. This expression demonstrates the uncertainty principle: to probe small-scale structure of an object, you need to have a scattering experiment with a high momentum transfer, because the Fourier transform averages out small-scale structure otherwise.

If the potential is central, *i.e.*, $V(\vec{x})$ is a function of $r = |\vec{x}|$ only. Then the expression Eq. (5) can be further simplified:

$$\begin{aligned} f^{(1)}(\vec{k'}, \vec{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d\cos\theta d\phi r^2 dr V(r) e^{iqr\cos\theta} \\ &= -\frac{m}{\hbar^2}\int_0^\infty dr r^2 V(r)\frac{e^{iqr} - e^{-iqr}}{iqr} \\ &= -\frac{2m}{\hbar^2}\frac{1}{q}\int_0^\infty dr r V(r)\sin qr. \end{aligned} \tag{6}$$

Therefore the scattering amplitude depends only on $q = |\vec{q}| = |\vec{k} - \vec{k'}| = 2k\sin(\theta/2)$. In other words, it is a function of the polar angle θ only $f(\vec{k'}, \vec{k}) = f(\theta)$. This is a statement independent of Born approximation.

2 Rutherford Scattering

2.1 Point Coulomb Source

One of the most important application of the Born approximation is to the Coulomb potential, because this is the relevant one for the Rutherford scattering experiment. By taking

$$V(r) = \frac{ZZ'e^2}{r}, \tag{7}$$

where I took the unit where $4\pi\epsilon_0 = 1$, we would like to calculate the differential cross section. Z is the charge of the scatterer (say, gold nucleus) and

Z' that of the incident particle (say, α particle). However, the expression Eq. (6) does not converge. Therefore, we start with a short-range potential called *Yukawa potential*

$$V(r) = V_0 \frac{e^{-\mu r}}{r}, \tag{8}$$

and take the limit $\mu \to 0$ to recover the Coulomb potential at the end of the calculations.[2] The Yukawa potential is a typical example of a short-ranged potential because it goes rapidly to zero once $r \gtrsim 1/\mu$. It is of great interest on its own apart from the limit $\mu \to 0$. The potential that binds protons and nucleons (nuclear force, or strong interaction) can be approximated by this type of potential, because the range of the nuclear force is only about 10^{-12} cm at most.

The formula Eq. (6) tells us that the scattering amplitude for the Yukawa potential Eq. (8) is

$$f(\theta) = -\frac{2mV_0}{\hbar^2} \frac{1}{q^2 + \mu^2}. \tag{9}$$

Different cross section is therefore given by

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\frac{2mV_0}{\hbar^2}\right)^2 \frac{1}{[2k^2(1-\cos\theta) + \mu^2]^2}. \tag{10}$$

The total cross section is obtained by integrating over $d\Omega = d\cos\theta d\phi$,

$$\sigma = \left(\frac{2mV_0}{\hbar^2}\right)^2 \frac{4\pi}{4k^2\mu^2 + \mu^4}. \tag{11}$$

We can now take the limit $\mu \to 0$ and $V_0 = ZZ'e^2$ to obtain results for the Coulomb potential,

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mZZ'e^2}{\hbar^2}\right)^2 \frac{1}{[2k^2(1-\cos\theta)]^2} = \frac{(2m)^2(ZZ'e^2)^2}{16(\hbar k)^4 \sin^4(\theta/2)}. \tag{12}$$

On the other hand, the total cross section Eq. (11) diverges! The divergence is in the $\cos\theta$ integral when $\theta \to 0$. In other words, the divergence occurs for the small momentum transfer $q \to 0$, which corresponds to large distances.

This result for the Coulomb scattering is exactly the same as in the classical theory by identifying $\hbar k$ as the momentum of the incident particle. It

[2] My normalization of V_0 is different from J.J. Sakurai by a factor of μ, so that $\mu \to 0$ limit is taken more easily.

is surprising that the Born approximation actually gives an exact result for the Coulomb potential, and it agrees with the classical calculation as well. This should be considered as a coincidence because there is no reason why any of them should come out to be the same.

The reason why the total cross section diverges is because the Coulomb potential is actually a *long-range* force. No matter how far the incident particles are from the charge, there is always an effect on the motion of the particles and they get scattered.

2.2 Form Factor

In practice, however, the total cross section cannot be infinite because the Coulomb potential by the gold nucleus is screened by the surrounding electrons in the gold atoms. What would be the cross section in that case? The Coulomb potential then is modified at long distances (distance beyond Bohr radius) where

$$V(\vec{x}) = \frac{ZZ'e^2}{|\vec{x}|} - \int d\vec{x'} \frac{Z'e^2}{|\vec{x} - \vec{x'}|}\rho(\vec{x'}), \tag{13}$$

where $\rho(\vec{x'})$ is the probability density of the electron cloud with the normalization $\int d\vec{x'}\rho(\vec{x'}) = Z$. $\rho(\vec{x'})$ is concentrated within the size of the atom $|\vec{x'}| \lesssim a$. Very far away from the atom, the second term cancels the first term and there is no potential.

Note that the second term is basically a convolution of the Coulomb potential and the probability density. Since the first Born amplitude is nothing but the Fourier transform of the potential, the convolution becomes a product of Fourier transforms, one for the Coulomb potential and the other for the probability density. Indeed, after performing the integral in Eq. (6), we find

$$f(\theta) = -\frac{2m}{\hbar^2}\frac{ZZ'e^2}{q^2}\left[1 - \frac{1}{Z}\int d\vec{x}\rho(\vec{x})e^{i\vec{q}\cdot\vec{x}}\right]. \tag{14}$$

In the limit $\vec{q} \to 0$, where the cross section diverges, two terms in the square bracket cancel because the second term approaches unity.

To gain more insight, let us take a simple case of the hydrogen atom $Z = 1$. The electron wave function in the ground state is

$$\psi(\vec{x}) = \frac{1}{\sqrt{4\pi}} 2a^{-3/2}e^{-r/a}. \tag{15}$$

$a = \hbar^2/me^2$ is the Bohr radius. The probability density of the electron cloud is then

$$\rho(\vec{x}) = |\psi(\vec{x})|^2 = \frac{1}{\pi a^3} e^{-2r/a}. \tag{16}$$

All we need to know now is the Fourier transform of this probability density. It is straightforward to obtain

$$\int d\vec{x} \rho(\vec{x}) e^{i\vec{q}\cdot\vec{x}} = \frac{16}{(4+q^2a^2)^2}. \tag{17}$$

For $\vec{q} \to 0$, the l.h.s. is simply the normalization of the wave function, *i.e.*, unity. The r.h.s. indeed gives the same limit. On the other hand, it vanishes when $q \gg a^{-1}$. In other words, for momentum transfer larger than the inverse size of the atom $\hbar/a$, the electron cloud does not change the cross section from the case of a point Coulomb source.

Eq. (14) is now given by

$$f(\theta) = -\frac{2m}{\hbar^2} Z' e^2 a^2 \frac{8 + 4(qa)^2}{(4+(qa)^2)^2}. \tag{18}$$

When $q \to 0$, the amplitude is regular and the total cross section converges. Recalling $q^2 = 2k^2(1-\cos\theta)$, we find

$$\sigma = \int d\Omega |f(\theta)|^2 = 2\pi \left(\frac{2m}{\hbar^2} Z' e^2 a^2\right)^2 \frac{-(k^2a^2) + 2\,(1+k^2a^2)\log(1+k^2a^2)}{k^2a^2 + k^4a^4} \tag{19}$$

For small $k \ll a^{-1}$, the last factor becomes unity, and the total cross section is

$$\sigma(k=0) = 2\pi \left(\frac{2m}{\hbar^2} Z' e^2 a^2\right)^2 = 8\pi Z'^2 \left(\frac{m}{m_e}\right)^2 a^2. \tag{20}$$

However, this result cannot be true. The geometric cross section of the target (the atom) is only of the order of πa^2. Because $m \gg m_e$, this total cross section is far larger than the geometric cross section. It signals the breakdown of perturbation theory: the Born approximation is invalid. Using the discussion of the validity in the next section, one can also see explicitly why that is the case. On the other hand, for a high momentum $k \gg a^{-1}$,

$$\sigma \simeq 8\pi Z'^2 \left(\frac{m}{m_e}\right)^2 a^2 \frac{2\log(1+k^2a^2) - 1}{k^2a^2}. \tag{21}$$

As long as $k \gg a^{-1}(m/m_e)$, Born approximation is valid and the total cross section can be trusted.

At much higher momentum transfers, the α-particle even starts to resolve the charge distribution of the nucleus

$$V(\vec{x}) = \int d\vec{x'} \frac{Z'e^2}{|\vec{x} - \vec{x'}|} \rho_N(\vec{x'}) - \int d\vec{x'} \frac{Z'e^2}{|\vec{x} - \vec{x'}|} \rho_e(\vec{x'}), \tag{22}$$

where $\rho_N(\vec{x'})$ is the charge distribution of the nucleus. Because at such high momentum transfer the second term is suppressed as seen above, the only important piece is the first term. Therefore the differential cross section reduces to the form

$$\frac{d\sigma}{d\Omega} = \left.\frac{d\sigma}{d\Omega}\right|_{\text{pointlike}} |F(q)|^2, \tag{23}$$

where the form factor $F(q)$ is nothing but the Fourier transform

$$F(q) = \frac{1}{Z} \int d\vec{x} \rho_N(\vec{x}) e^{i\vec{q}\cdot\vec{x}}. \tag{24}$$

In fact, Rutherford experiment already showed the deviation from the point-like Coulomb source at high momentum transfer (large angle scattering), which led him to estimate the size of the nucleus.

Fig. 1 shows the form factor $|F(q)|^2$ in an electron-nucleus scattering experiment. The oscillatory behavior can be understood qualitatively in the following way. Imagine a sphere of radius a with a uniform charge density ρ_0 such that $Z = \frac{4\pi}{3} a^3 \rho_0$. The form factor, the Fourier transform, is given by

$$\begin{aligned} F(q) &= \int d\vec{x} \rho_N(\vec{x}) e^{i\vec{q}\cdot\vec{x}} && (25) \\ &= \int_0^{2\pi} d\phi \int_{-1}^{1} d\cos\theta \int_0^a r^2 dr \rho_0 e^{iqr\cos\theta} && (26) \\ &= 2\pi \int_0^a r^2 dr \rho_0 \frac{e^{iqr} - e^{-iqr}}{iqr} && (27) \\ &= 4\pi \frac{\sin aq - aq\cos aq}{q^3} \rho_0. && (28) \end{aligned}$$

Overall, this function goes down as $1/q^2$ at large q, while it oscillates in the numerator. It oscillates because the Fourier transform depends sensitively on how many waves fit inside the nucleus. The true charge density distribution is not sharply cutoff as a uniform sphere, but somewhat smoothed out at the edge, but still similar. Fourier transform of the measured form factor determined the true charge density distribution inside the nucleus, as seen in Fig. 2

Later, much more precise and higher energy electron-proton scattering experiments were performed, which showed that the form factor has an approximate dipole form (Fig. 3)

$$F(q) \simeq \frac{1}{(1+q^2 a_N^2)^2}, \tag{29}$$

where $a_N \simeq 0.26$ fm. From the inverse Fourier transform, one can see that the charge density of the proton has approximately an exponential profile $\propto e^{-r/a_N}$. This is probably one of the earliest evidences for the composite nature of the proton.

3 Born Expansion

Of course, the first Born approximation is only the leading order in V. We can work out higher orders from Eq. (3), by iteratively insert the r.h.s. of the equation at a given order in V back into the $|\psi\rangle$. We then have the infinite series

$$\begin{aligned} |\psi\rangle = |\phi\rangle &+ \frac{1}{E-H_0+i\epsilon} V|\phi\rangle + \frac{1}{E-H_0+i\epsilon} V \frac{1}{E-H_0+i\epsilon} V|\phi\rangle \\ &+ \frac{1}{E-H_0+i\epsilon} V \frac{1}{E-H_0+i\epsilon} V \frac{1}{E-H_0+i\epsilon} V|\phi\rangle + \cdots . \end{aligned} \tag{30}$$

This is called Born expansion, and the Born approximation we used is nothing but the first term in this systematic expansion. The physical meaning of this equation is obvious. The first term is the wave which did not get scattered. The second term is the wave that gets scattered at a point in the potential and then propagates outwards by the $1/(E-H_0+i\epsilon)$ operator. In the third term, the wave gets scattered at a point in the potential, propagates for a while, and gets scattered again at another point in the potential, and propagates outwards. In the $n+1$-th term, there are n times scattering of the wave before it propagates outwards.

More formally, an operator called T-matrix is used often in scattering problems. The definition is

$$V|\psi\rangle = T|\phi\rangle. \tag{31}$$

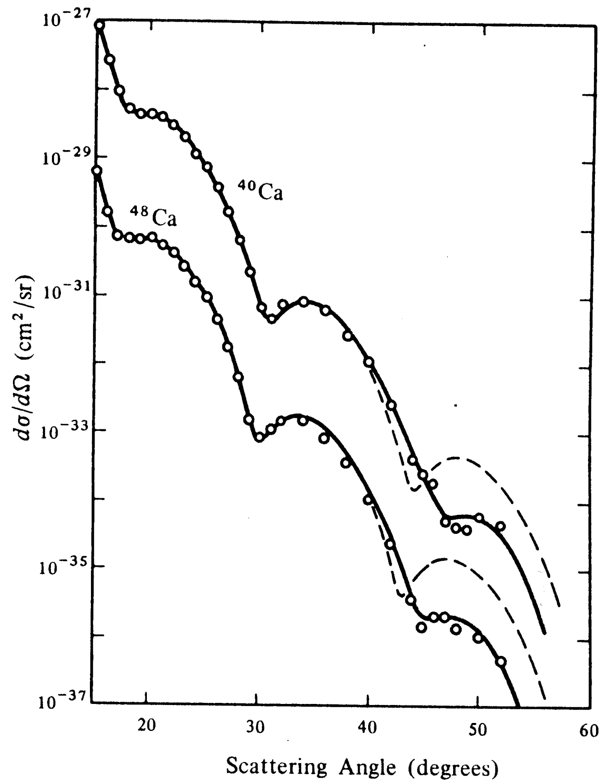

Fig. 6.5. Differential cross section for scattering of 750-MeV electrons from calcium isotopes. The cross section for ^{40}Ca has been multiplied by a factor of 10, and that for ^{48}Ca by 10^{-1}. [From J. B. Bellicard et al., *Phys. Rev. Letters* **19**, 527 (1967).]

Figure 1: Taken from *Subatomic Physics*, Hans Frauenfelder and Ernest M. Henley, Prentice-Hall, Inc., 1974.

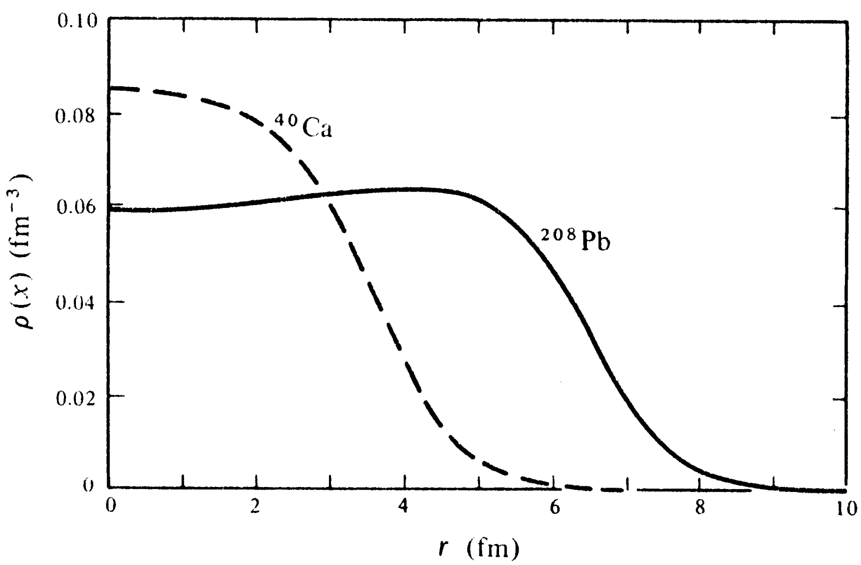

Fig. 6.7. Probability distribution for ^{40}Ca and ^{208}Pb, obtained by electron scattering. (Courtesy of D. G. Ravenhall.)

Figure 2: Taken from *Subatomic Physics*, Hans Frauenfelder and Ernest M. Henley, Prentice-Hall, Inc., 1974.

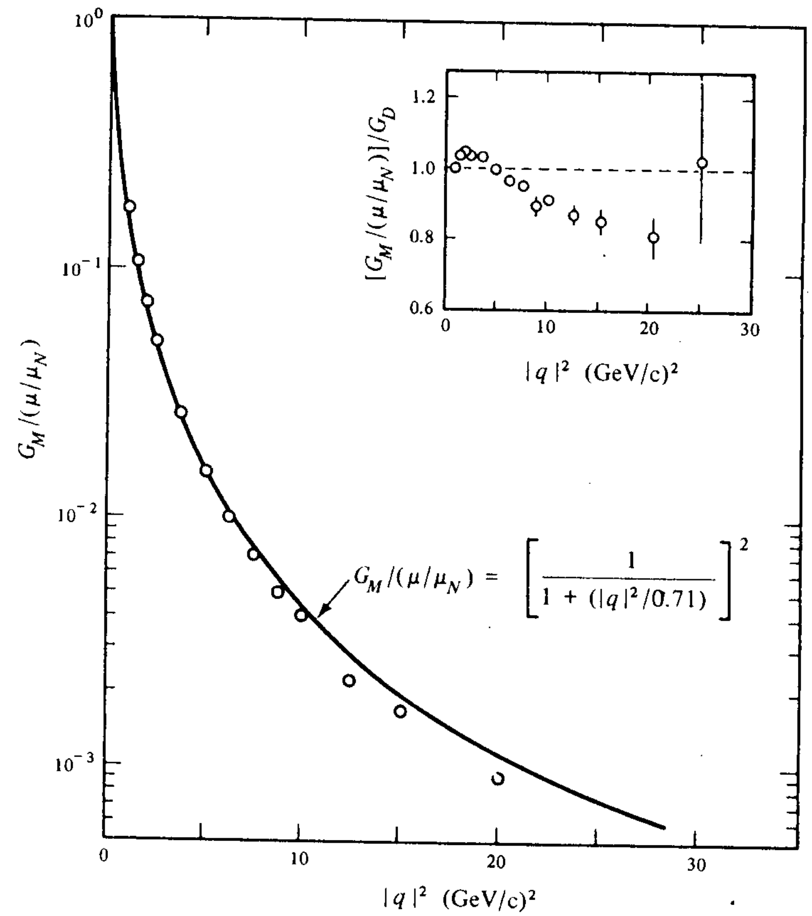

Fig. 6.13. Values of the proton magnetic form factor G_M, normalized by division with the proton magnetic moment, plotted versus the momentum transfer q^2. An empirical *dipole fit* to the data is shown as a solid line. The insert shows the ratio of the measured values of G_M to the dipole fit. From P. N. Kirk et al., *Phys. Rev.* **D8**, 63 (1973).

Figure 3: Taken from *Subatomic Physics*, Hans Frauenfelder and Ernest M. Henley, Prentice-Hall, Inc., 1974.

We always take $|\phi\rangle = |\hbar\vec{k}\rangle$. This seemingly weird definition is actually useful as seen below. The scattering amplitude derived in the lecture note "Scattering Theory I" is

$$f(\vec{k'},\vec{k}) = -\frac{(2\pi)^3}{4\pi}\frac{2m}{\hbar^2}\langle\hbar\vec{k'}|V|\psi\rangle. \tag{32}$$

Using the definition of the T-matrix, we find

$$f(\vec{k'},\vec{k}) = -\frac{(2\pi)^3}{4\pi}\frac{2m}{\hbar^2}\langle\hbar\vec{k'}|T|\hbar\vec{k}\rangle. \tag{33}$$

Hence, the T-matrix element has a physical interpretation of the transition (hence T) from the initial momentum $\hbar\vec{k}$ to the final momentum $\hbar\vec{k'}$.

Using the Lippmann–Schwinger equation Eq. (1), and multiplying the both sides by V from left, we find

$$T|\phi\rangle = V|\phi\rangle + V\frac{1}{E-H_0+i\epsilon}T|\phi\rangle, \tag{34}$$

and hence

$$T = V + V\frac{1}{E-H_0+i\epsilon}T. \tag{35}$$

In other words, a formal solution to the T-matrix is

$$T = \frac{1}{1-V\frac{1}{E-H_0+i\epsilon}}V. \tag{36}$$

By Taylor expanding this operator in geometric series, we find

$$T = V + V\frac{1}{E-H_0+i\epsilon}V + V\frac{1}{E-H_0+i\epsilon}V\frac{1}{E-H_0+i\epsilon}V + \cdots. \tag{37}$$

This proves the Born expansion Eq. (30).

In the coordinate space, for example, the second Born term is given by

$$\begin{aligned}&\langle\vec{x}|\frac{1}{E-H_0+i\epsilon}V\frac{1}{E-H_0+i\epsilon}V|\phi\rangle \\ &= \int d\vec{x'}d\vec{x''}\frac{-2m}{\hbar^2}\frac{e^{ik|\vec{x}-\vec{x'}|}}{4\pi|\vec{x}-\vec{x'}|}V(\vec{x'})\frac{-2m}{\hbar^2}\frac{e^{ik|\vec{x'}-\vec{x''}|}}{4\pi|\vec{x'}-\vec{x''}|}V(\vec{x''})\phi(\vec{x''}),\end{aligned} \tag{38}$$

where $\phi(\vec{x''}) = e^{i\vec{k}\cdot\vec{x''}}/(2\pi\hbar)^{3/2}$.

4 Validity of Born Approximation

Born approximation replaces ψ by ϕ in Lippmann–Schwinger equation, which is integrated together with the potential. Therefore, in order for Born approximation to be good, the difference between ψ and ϕ must be small where the potential exists. The self-consistency requires that

$$|\psi(\vec{x}) - \phi(\vec{x})| \ll |\phi(\vec{x})| \tag{39}$$

where $V(\vec{x})$ is sizable, and the l.h.s. can be evaluated within Born approximation itself. From Lippmann–Schwinger equation (the one before taking the limit of large r), we find

$$\left| \frac{2m}{\hbar^2} \int d\vec{x'} \frac{e^{ik|\vec{x}-\vec{x'}|}}{4\pi|\vec{x}-\vec{x'}|} V(\vec{x'}) e^{i\vec{k}\cdot\vec{x'}} \right| \ll 1. \tag{40}$$

In particular, we require this condition at $\vec{x} = 0$ where the potential is the strongest presumably.

You discussed delta-function potential $V = \gamma\delta(\vec{x})$ in the homework problem and showed that it actually does not cause any scattering in three dimensions. The Born approximation Eq. (5), however, suggests a scattering amplitude

$$f^{(1)}(\vec{k'}, \vec{k}) = -\frac{1}{4\pi} \frac{2m}{\hbar^2} \gamma. \tag{41}$$

This is in contradiction to the exact result. In this case, the validity condition Eq. (40) is indeed violated because

$$\left| \frac{2m}{\hbar^2} \int d\vec{x'} \frac{e^{ik|\vec{x}-\vec{x'}|}}{4\pi|\vec{x}-\vec{x'}|} \gamma\delta(\vec{x'}) e^{i\vec{k}\cdot\vec{x'}} \right| = \frac{2m}{\hbar^2} \frac{1}{4\pi|\vec{x}|} |\gamma| \tag{42}$$

can be arbitrarily large when $|\vec{x}| \to 0$. In fact, the second Born amplitude diverges non matter how small γ is. Inserting $V = \gamma\delta(\vec{x})$, Eq. (38) is

$$\begin{aligned}
&\langle\vec{x}| \frac{1}{E - H_0 + i\epsilon} V \frac{1}{E - H_0 + i\epsilon} V |\phi\rangle \\
&= \int d\vec{x'} d\vec{x''} \frac{-2m}{\hbar^2} \frac{e^{ik|\vec{x}-\vec{x'}|}}{4\pi|\vec{x}-\vec{x'}|} \gamma\delta(\vec{x'}) \frac{-2m}{\hbar^2} \frac{e^{ik|\vec{x'}-\vec{x''}|}}{4\pi|\vec{x'}-\vec{x''}|} \gamma\delta(\vec{x''}) \phi(\vec{x''}) \\
&= \left(\frac{-2m}{\hbar^2}\right)^2 \gamma^2 \frac{e^{ik|\vec{x}|}}{4\pi|\vec{x}|} \frac{e^{ik|\vec{0}|}}{4\pi|\vec{0}|} = \infty.
\end{aligned} \tag{43}$$

Clearly, Born expansion is not appropriate for this potential, and hence your result from the HW #1 is not inconsistent with an apparent finite scattering amplitude from the Born approximation.

For a smooth central potential, with a magnitude of order V_0 and a range of order a, we can qualitatively work out the validity constraint Eq. (40). Taking $\vec{k}$ along the z axis, and looking at $\vec{x} \simeq 0$ where the potential is most important presumably (and relabeling $\vec{x'}$ as $\vec{x}$), the condition is

$$\frac{2m}{\hbar^2}\left|\int d\vec{x}\frac{e^{ikr}}{4\pi r}V(\vec{x})e^{ikz}\right| \ll 1. \tag{44}$$

When $k \ll a^{-1}$, we can ignore the phases in the integral, and it is given roughly by

$$\frac{2m}{\hbar^2}|V_0|a^2\frac{1}{2} \ll 1 \qquad (k \ll a^{-1}). \tag{45}$$

Numerical coefficients are not to be trusted. On the other hand, when $k \gg a^{-1}$, the phase factor oscillates rapidly and we can use stationary phase approximation. The exponent is $ikr + ikz$, and it is stationary only along the negative z-axis $z = -r$. Expanding around this point, it is $ikr + ikz = ik(x^2 + y^2)/r + O(x^3, y^3)$. The Gaussian integral over x, y then gives a factor of $\pi r/k$, while z is integrated along the stationary phase direction from $-a$ to 0. Therefore, the validity condition is given roughly by

$$\frac{2m}{\hbar^2}\frac{a}{4k}|V_0| \ll 1 \qquad (k \gg a^{-1}). \tag{46}$$

On the other hand, we can estimate the total cross section in both limits. The scattering amplitude in the Born approximation Eq. (5) is

$$\begin{aligned} f^{(1)}(\vec{k'}, \vec{k}) &= -\frac{1}{4\pi}\frac{2m}{\hbar^2}\int d\vec{x}V(\vec{x})e^{i\vec{q}\cdot\vec{x}} \\ &\sim -\frac{1}{4\pi}\frac{2m}{\hbar^2}V_0\frac{4\pi}{3}a^3 \qquad (q \ll a^{-1}). \end{aligned} \tag{47}$$

For a large momentum transfer, say along the x axis, y and z integral each gives a factor of a because of no phase variation, while x integral oscillates rapidly and cancels mostly; it leaves only $\sim 1/q$ contribution from non-precise cancellation. Therefore,

$$f^{(1)}(\vec{k'}, \vec{k}) \sim -\frac{1}{4\pi}\frac{2m}{\hbar^2}V_0\frac{\pi a^2}{q} \qquad (q \gg a^{-1}). \tag{48}$$

Because the momentum transfer q is of the order of k (except the very forward region which we neglect from this discussion), the total cross sections are roughly

$$\sigma \sim \begin{cases} \frac{1}{4\pi}\left(\frac{2m}{\hbar^2}V_0\frac{4\pi}{3}a^3\right)^2 & (k \ll a^{-1}) \\ \frac{1}{4\pi}\left(\frac{2m}{\hbar^2}V_0\frac{\pi a^2}{q}\right)^2 & k \gg a^{-1}). \end{cases} \tag{49}$$

It is interesting to note that, once the validity condition Eqs. (45,46) is satisfied, the total cross section is always smaller than the geometric cross section $4\pi a^2$.

$$\sigma \ll \frac{16}{9}\pi a^2 \qquad (k \ll a^{-1}) \tag{50}$$

$$\sigma \ll 4\pi a^2 \qquad (k \gg a^{-1}). \tag{51}$$

If you find a Born cross section larger than the geometric cross section, you should be worried.