221B Lecture Notes
Many-Body Problems II
Nuclear Physics

1 Nuclei

Nuclei sit at the center of any atoms. Therefore, understanding them is of central importance to any discussions of microscopic physics. Due to some reason, however, the nuclear physics had not been taught so much in the standard physics curriculum. I try to briefly review nuclear physics in about a week. Obviously I can't go into much details, but hope to give you at least a rough idea on nuclear physics.

As you know, nuclei are composed of protons and neutrons. The number of protons is the atomic number Z, and the mass number A is approximately the total number of *nucleons*, a collective name for protons and neutrons. Therefore

$$A = N + Z \tag{1}$$

where N is the number of neutrons. We know that nuclei are very small. An emperical formula for the size of the nuclei, which can be measured using the form factor in elastic electron-nuclei scattering, is

$$R = r_0 A^{1/3}, \qquad r_0 = 1.12 \text{ fm}. \tag{2}$$

This is a good approximation practically for all nuclei with $A \gtrsim 12$. Here, fm $= 10^{-13}$cm, or sometimes called also "Fermi" rather than femto-meter, and the nuclei are smaller by five orders of magnitude than the atoms. What the formula means is that the nuclear density is more-or-less constant for any nuclei, $\rho = 1.72 \times 10^{38}$ nucleons/cm^3 $= 0.172$ nucleons/fm^3. Of course, the nuclear density does not drop to zero abruptly. The form factor measurement is often fitted to the size and the "surface thickness," within which the density smoothly falls from the constant to zero. The result is that the surface thickness is about $t \simeq 2.4$ fm.

2 Emperical Mass Formula

Gross properties of nuclei are manifested in the empirical (or Weizsäcker) mass formula. Recall Einstein's relation $E = mc^2$, which tells us that the to-

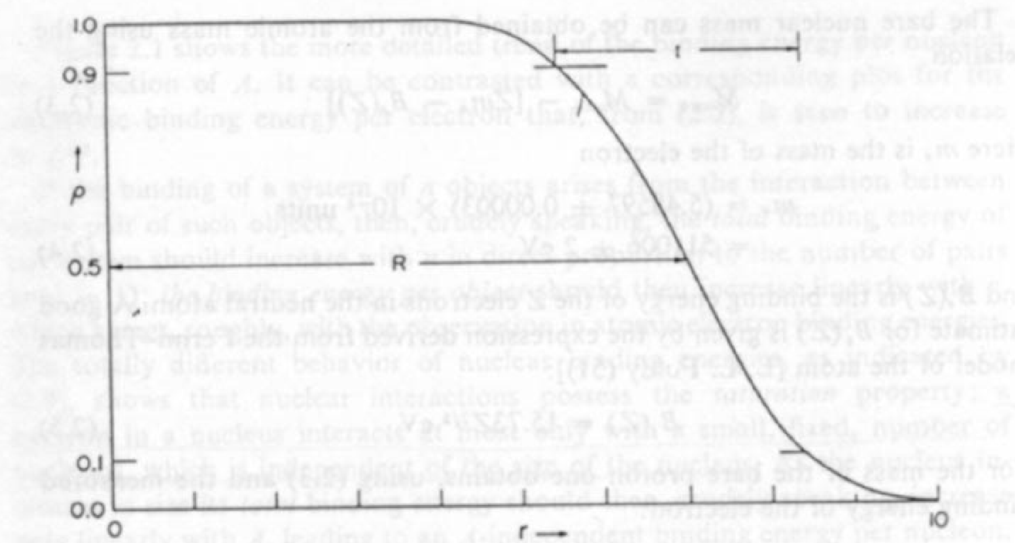

FIG. 1.2. Typical charge density. *R* is the nuclear radius and *t* is the surface thickness.

Figure 1: From "Theoretical Nuclear Physics," by Amos deShalit and Herman Feshbach, New York, Wiley, 1974.

tal mass of nuclei has information on its composition as well as its interaction energies. The empirical mass formula is

$$m_{\text{nucleus}}(Z,N) = Zm_p + Nm_n - \frac{B}{c^2}, \tag{3}$$

where the last term is the "mass deficit" due to the binding energy B, and is given by

$$B = a_v A - a_s A^{2/3} - a_{\text{sym}} \frac{(Z-N)^2}{A} - a_C \frac{Z^2}{A^{1/3}} + \delta(A). \tag{4}$$

Among all these terms, the first terms is the most important one, giving roughly constant binding energy per nucleon. If you neglect all the other terms, the binding energy is roughly 8.5 MeV/nucleon. But if you fit the data with all other terms, the number of course comes out differently. We discuss each of the terms below.

The first term is called the volume term with $a_v = 15.68$ MeV, representing that the total binding energy is roughly proportional to the number of nucleons. This is the dominant term in the formula. Other terms show the variation of the binding energy as a function of N and Z. The second term is called the surface term with $a_s = 18.56$ MeV, representing that the binding energy is lost somehow proportional to the surface area. These two terms can be qualitatively explained by the so-called liquid drop model of nuclei. You can view a nucleus as a tightly packed drop of nucleons, each feeling attractive force from its neighbors. The point is that the force comes basically only from its neighbors due to the short-ranged nature of the nuclear force responsible for binding nuclei. Because the number of "neighbors" is basically the same for any nucleon given the constant nuclear density we've seen above, the amount of binding energy is proportional to the number of nucleons, giving rise to the volume term. This is said to be "saturation" of nuclear binding, and the nucleons basically don't *see* nucleons beyond their neighbors. But those at the surface receive less binding because they do not have about a half of neighbors. The loss of the binding energy is given by the surface term.

The symmetry term is less obvious. The empirical fact is that stable nuclei require more-or-less the same number of protons and neutrons, especially true for light nuclei. Think about common nuclides: ^{4}He, ^{12}C, ^{14}N, ^{16}O, etc, with high natural abundances. This point will be understood in terms of "Fermi gas" model of nuclei. By putting in neutrons and protons as free particles

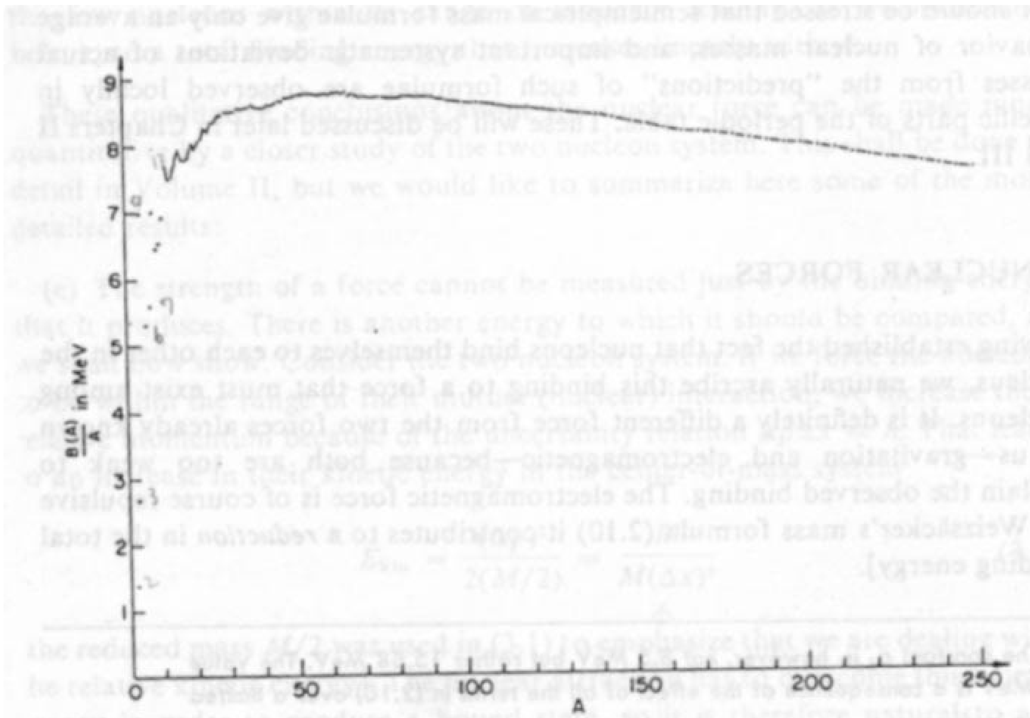

FIG. 2.1. Binding energy per nucleon as a function of *A*.

Figure 2: Nuclear binding energy is more-or-less independent of its size, roughly about 8.5 MeV/nucleon. The first few peaks are for ^{4}He, ^{12}C, ^{16}O. The maximum is for ^{56}Fe. From "Theoretical Nuclear Physics," by Amos deShalit and Herman Feshbach, New York, Wiley, 1974.

in a Fermi-degenerate gas, protons and neutrons fill up levels independently, and it is energetically favorable to keep the Fermi energies for protons and neutrons the same for a given total number of nucleons (mass number). The symmetry term, with $a_{Sym} = 28.1$ MeV, reflects the rise in the energy when they are not equal with a parabolic approximation around the minimum $Z = N$.

The Coulomb term has the obvious meaning of total Coulomb energy among protons (neutrons are electrically neutral!). Because the number of protons is Z, and there is Coulomb potential between any pairs of protons (long-ranged force unlike the nuclear binding force), the energy goes as Z^2. The typical distance among them is the nuclear size, given by $A^{1/3}$, hence the dependence $Z^2/A^{1/3}$, with the coefficient $a_C = 0.717$ MeV. It shows that the Coulomb interaction is actually a very weak interaction compared to the nuclear force. Of course, the actual size of the Coulomb energy can be important especially for large nuclei, because it grows like Z^2 (even if you scale Z and A together, it grows as $Z^{5/3}$). This tends to prefer smaller Z for a given A. The competition of the Coulomb term and the symmetry term gives a preferred fraction of protons for a given A, which becomes smaller and smaller as A increases, consistent with the observed band of stable isotopes.

Finally the last term is called the pairing term. There is a tendency that nucleons want to be *paired* between a given state and its time-reversed state, *i.e.*, the opposite orbital and spin angular momenta. Because of this property, even-even nuclei (nuclei with even number of protons and even number of neutrons) have all 0^+ ground state. There is a sizable difference in the binding energies between nuclei with all nucleons paired (even-even ones) and those with some nuclei unpaired (even-odd, odd-even, and odd-odd). The pairing term represents the energy difference among them, given by

$$\delta(A) = \begin{cases} 34A^{-3/4} & \text{MeV for odd-odd nuclei} \\ 0 & \text{MeV for odd-even nuclei} \\ -34A^{-3/4} & \text{MeV for even-even nuclei} \end{cases} . \qquad (5)$$

Looking at the plot Fig. (2), there are a few anomalously high binding energies for low A. They are ^{4}He, ^{12}C, ^{16}O. The maximum is for ^{56}Fe. The presence of a maximum means that any thermonuclear fusion process, such as stellar burning, cannot produce nuclei beyond iron. When an iron nucleus tries to fuse with further nuclei, it has to *acquire* energy to do so rather than releasing the binding energy. Some chemical elements were formed when the

Universe was only about a second old. It was a hot dense plasma of mostly electrons, positrons, photons and neutrinos. As the Universe expands the plasma cools, and eventually the neutrons either decay or get sucked into ^{4}He nuclei. Some deutrium (^{2}H), ^{3}He, ^{6}Li, ^{7}Li were also formed. All heavier elements are products of stellar burning. The Sun is burning mostly by fusing hydrogen nuclei (*i.e.*, protons) into ^{4}He. When the Sun gets old and uses up hydrogen, the core pressure from the fusion energy suddenly drops and it can no longer support its entire mass. Then the core contracts while the outer region expands dramatically. The Sun becomes a red giant. As the core contracts, the temperature rises and ^{4}He start to fuse despite its larger Coulomb barrier than protons. Where the process stops depends on the mass of the star. The Sun probably stops mostly with ^{12}C, ^{14}N, and ^{16}O. As the fusion process winds down, the core pressure weakens again, the outer region gets blown off while the core further collapses. The collapse stops when the electron degeneracy pressure becomes important, and the Sun becomes a white dwarf, only slowly burning ^{4}He and becomes rather dim. Heavier stars can further burn C, N, O, eventually going all the way to iron. However, as we have seen, iron cannot fuse any further, and then there is no way to support the entire mass of the star, and even the electron degeneracy pressure wouldn't be enough for those. As the core collapses, the entire star basically becomes a single nucleus: a neutron star. The Coulomb repulsion among the protons favors the conversion of protons to neutrons by absorbing electrons and emitting neutrinos. The whole star then is supported by the hard core repulsion in the nuclear force (see the next section) and neutron degeneracy pressure. The bounce from the collapse results in a supernovae, giving rise to a highly dynamic condition in the envelope. It is hoped that the exploding envelope synthesizes elements beyond iron, producing elements such as silver, gold, lead, platinum, uranium, thorium. The idea is that iron (and eventually heavier elements) keep sucking in neutrons without Coulomb barrier, and become higly neutron-rich nuclei. They decay into more balanced nuclei by beta-decay. This is called nuclear r-process (r for rapid).[1]

[1]Steve Boggs in our Department is trying to verify the supernovae as sites for nuclear r-process by observing X-ray and gamma-ray from supernova remnants.

3 Nuclear Force

Protons and neutrons are bound inside nuclei, despite the Coulomb respulsion among protons. Therefore there must be a different and much stronger force acting among nucleons to bind them together. This force is called nuclear force, nuclear binding force, or in more modern settings, *the strong interaction*. (Here, we are not talking about *a* strong interaction. This is the name of the force.) Here are notable properties of the nuclear binding force.

1. It is much stronger than the electromagnetic force. In the empirical mass formula, we saw that the coefficient of the Coulomb term is more than an order of magnitude smaller than the other terms in the binding energy.

2. It is an attractive force, otherwise nucleons wouldn't bind.

3. It is short-ranged, acts only up to 1–2 fm.

4. It has the saturation property, giving nearly constant $B/A \simeq 8.5$ MeV. This is in stark contrast to the electromagnetic force. For instance, the Thomas–Fermi model of atoms gives $B = 15.73 Z^{7/3}$ eV that grows with a very high power in the number of particles.

5. The force depends on spin and charge states of the nucleon. To understand nuclei and nucleon-nucleon scattering data, we need not only a potential $V(r)$ between nucleons in the Hamiltonian but also the spin-spin term $\vec{\sigma}_1 \cdot \vec{\sigma}_2 V(r)$, the spin-orbit term $(\vec{\sigma}_1 + \vec{\sigma}_2) \cdot \vec{L} V(r)$, and the tensor term $[3(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r}) - r^2 \vec{\sigma}_1 \cdot \vec{\sigma}_2] V(r)$.

6. It can exchange charge. If you do neutron-proton scattering experiment, you not only see a forward peak but also a backward peak. Note that a forward peak is analogous to a large impact parameter in the classical mechanics where there is little deflection (recall Rutherford scattering), and exists for pretty much any scattering processes. But a backward peak is quite unusual. The interpretation is that when the proton appears to be backscattered, it is actually a neutron which converted to a proton because of the nuclear reaction. In other words, the neutron is scattered to the forward angle, but has converted to proton by the scattering and we are fooled to see the proton scattered backward. This is the charge-exchange reaction.

7. Even though the nuclear force is attractive to bind nucleons, there is a repulsive core when they approach too closely, around 0.5 fm. They basically cannot go closer.

8. The nuclear force has "charge symmetry," which means that we can make an overall switch between protons and neutrons without changing forces among them. For instance, nn and pp scattering are the same (except for the obvious difference due to the electric charge). For example, "mirror nuclei," which are related by switching protons and neutrons, have very similar excitation spectra. Examples include ^{13}C and ^{13}N, ^{17}O and ^{17}F, etc.

9. A stronger version of the charge symmetry is "charge independence." Not only nn and pp scattering are the same, but also np scattering is also the same under the "same configuration" which I specify below using the concept of isospin.

The last item needs some more explanations. There is a new symmetry in the nuclear force called isospin, proposed originally by Heisenberg. The idea is very simple: regard protons and neutrons as identical particles. But of course, you can't; they are different particles, right? They even have different massees! Well, the trick is to introduce a new quantum number, isospin, which takes values $+1/2$ and $-1/2$ just like the ordinary spin. We say a proton is a nucleon with $I_z = +1/2$, while a neutron with $I_z = -1/2$. At this point, it is just semantics. But the important statement is this: the nuclear force is invariant under the isospin rotation, just like the Hamiltonian of a ferromagnet is invariant under the rotation of spin. Then you can classify states according to the isospin quantum numbers because the nuclear force preserves isospin. But what about the mass difference, then? The point is that their masses are actually quite similar: $m_p = 938.3$ MeV$/c^2$ and $m_n = 939.6$ MeV$/c^2$. To the extent that we ignore the small mass difference, we can treat them identical. Another question is the obvious difference in their electric charges $+|e|$ and 0. Again, the Coulomb force is not the dominant force in nuclei, as we have seen in the empirical mass formula. We can ignore the difference in the electric charge and put it back in as a "small" perturbation.

The charge symmetry is a limited example of the isospin invariance. It corresponds to the overall reversal of all isospins. If you reverse all spins s_z, that is basically the 180° rotation around the y-axis, and you obtain another

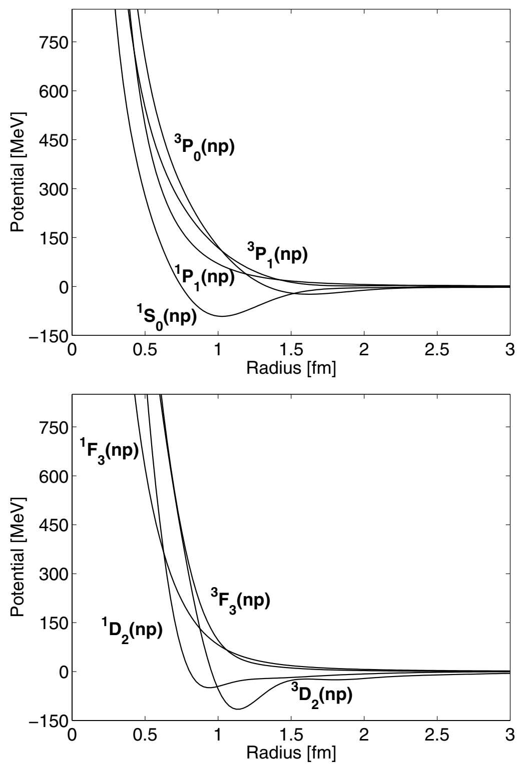

Figure 3: A recent analysis of nucleon-nucleon scattering data to obtain the nucleon-nucleon potential. Taken from A. Funk, H. V. von Geramb, K. A. Amos, nucl-th/0105011.

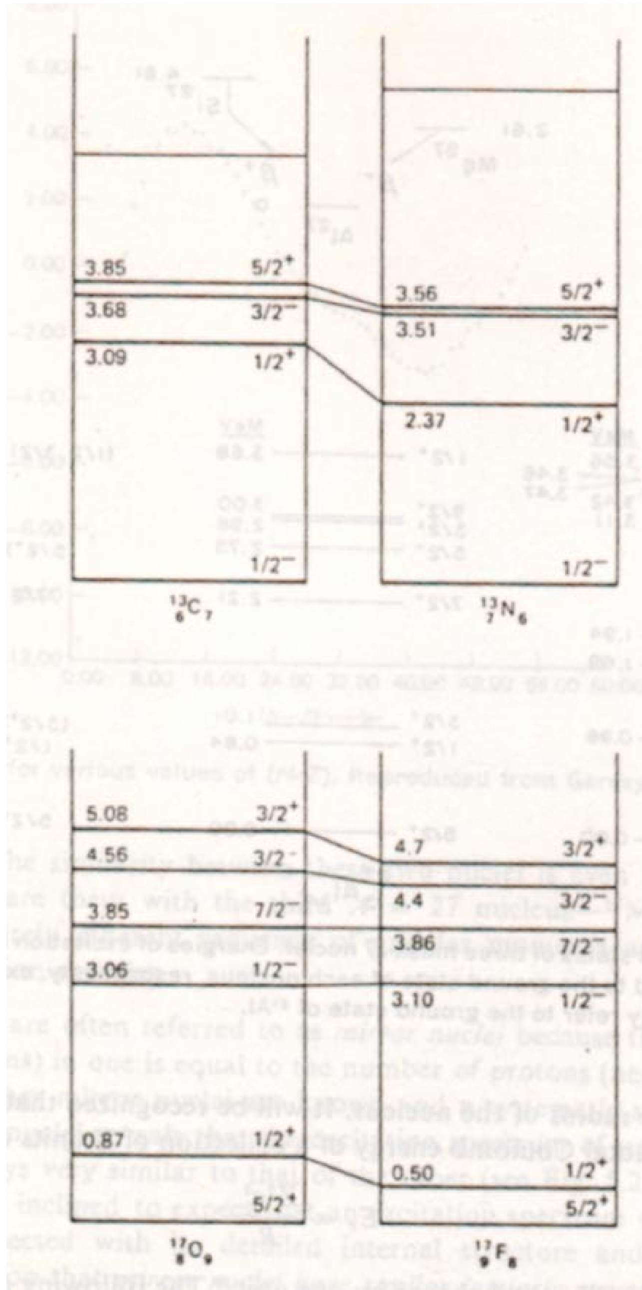

Figure 4: Comparison of excitation spectrum of two mirror nuclei, ^{13}C and ^{13}N, ^{17}O and ^{17}F. From "Theoretical Nuclear Physics," by Amos deShalit and Herman Feshbach, New York, Wiley, 1974.

state with degenerate energy. Likewise, if you revsere all isospins, by rotating the isospin around the "isospin y-axis" by 180°, you interchange protonts with neutrons, just like interchanging spin up and spin down states. If the nuclear force is indeed invariant under the isospin rotation, it must also be invariant under the isospin reversal. Fig. 4) shows that indeed the nuclear spectra approximately respect this invariance. Of course, isospin is not an exact symmetry because protons and neutrons have different electric charges. But the isospin invariance goes even further ("charge independence"). It says that the not only the interaction between *pp* and *nn* are the same ("charge symmetry"), also *np* is, except that you have to carefully select the configuration. Here is what is required. Because proton and neutron both carry $I = 1/2$ (and opposite $I_z = \pm 1/2$), two nucleon states would have both $I = 1$ and $I = 0$ components. Both *pp* and *nn* states are said to be in the $I = 1$ state. On the other hand, the *np* state can either be in the $I = 1$ or $I = 0$ states. But the fermion wave function must be anti-symmetric while $I = 1$ ($I = 0$) isospin wave function is symmetric (anti-symmetric). Therefore, if the space and spin wave function of a *np* state is symmetric (anti-symmetric), it selects $I = 0$ ($I = 1$) isospin wave function. This way, you can separate purely $I = 1$ part of the *np* wave function, and compare the interaction to that of the *nn* and *pp* states. And they are indeed the same up to corrections from Coulomb interaction. On the other hand, the force in the $I = 0$ state can be different. For instance, the only two-nucleon bound state is the deuterium, an *np* state. What is suggests is that the bound state is in the $I = 0$ state, and anti-symmetric isospin wave function. Then the rest of the wave function must be symmetric. For a given potential, the S-wave is always more binding than the P-wave just because it lacks the centrifugal barrier. Therefore the deuterium is likely to be in the S-wave, a symmetric spatial wave function. Then the spin wave function must be symmetric, $S = 1$. Indeed deuterium does have spin one. A more quantitative test can be seen in Fig. 5. ^{21}F, ^{21}Ar, ^{21}Na, and ^{21}Mg all have the mass number 21. Assuming ^{18}F is in the $I = 0$ state, all four nuclei can be obtained by adding three neutrons to it, which can be in either $I = 3/2$ or $I = 1/2$ states. The nuclear excitation spectra show states common only between ^{21}Ar and ^{21}Na, which are in the $I = 1/2$ state, or states common to all four of them, which are in the $I = 3/2$ state. Similary check can be done among ^{14}C, ^{14}N, ^{14}O, which show states common to all of them ($I = 1$) or states special to ^{14}N ($I = 0$).

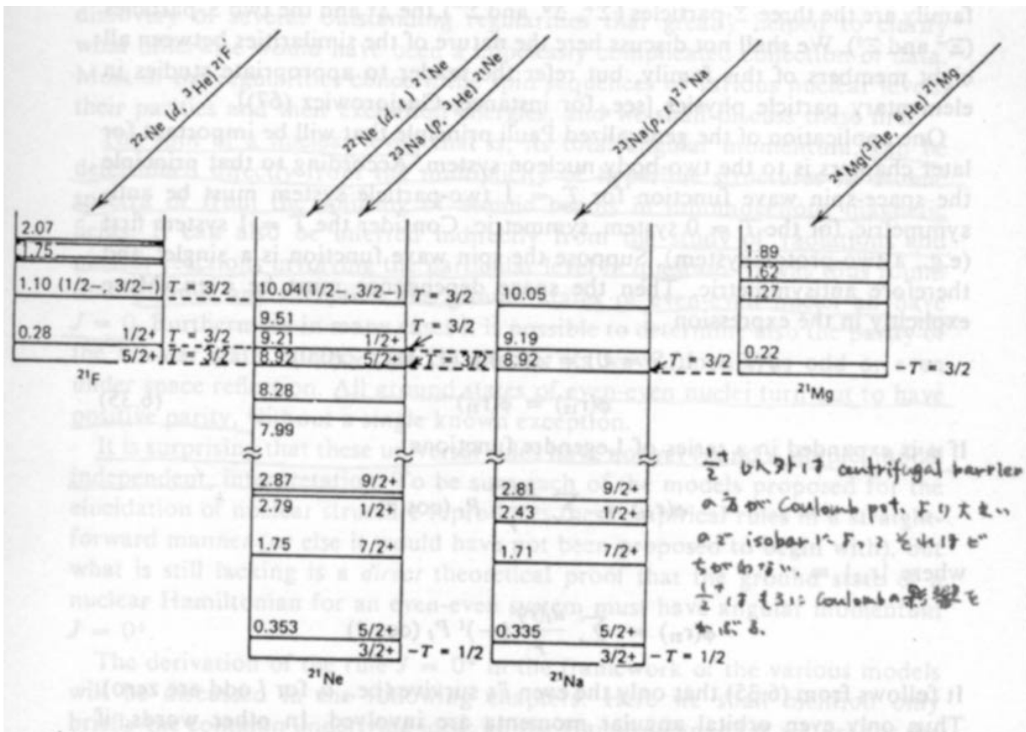

FIG. 6.1. Energy level diagrams for the members of the mass-21 isospin quartet showing the positions of the $T = 3/2$ levels in each nucleus. For clarity, the ground state energies of mirror-nuclei have been equated, and many of the excited levels of Ne and Na below 10-MeV excitation have been deleted [Butler, Cerny, and McCarthy, (68)].

Figure 5: Comparison of excitation spectrum of four nuclei with the same mass number, showing states with $I = 1/2$ and $I = 3/2$ multiplet structure. From "Theoretical Nuclear Physics," by Amos deShalit and Herman Feshbach, New York, Wiley, 1974.

4 Yukawa Theory and Two-nucleon System

Given the properties of the nuclear force described in the previous section, what, after all, is it? I briefly go through the explanations in a quasi-historic way, but this is by no means rigorous or exhaustive. But hopefully I can give you an idea on how we came up with the current understanding, namely Quantum ChromoDynamics (QCD).

The obvious oddity with the nuclear force was its short-rangedness. People knew gravity and electromagnetism; both of them are long-ranged, with their potential decreasing as $1/r$. On the other hand, the nuclear force is practically zero beyond a few fm. As we will discuss in the "Quantization of Radiation Field," the electromagnetic interaction is described by photons in the fully quantum theory. Likewise, the nuclear force must also involve a particle that is responsible for the force. Such a particle is often called a "force carrier." The idea of the force carrier is simple: quantum mechanics allows you to "borrow" energy ΔE violating its conservation law as long as you give it back within time $\Delta t \sim \hbar/\Delta E$ allowed by the uncertainty principle. Take the case of an electromagnetic reaction, say electron proton scattering. An electron cannot emit a photon by itself because that would violate energy and momentum conservation. But it can do so by "borrowing" energy as long as the created photon is absorbed by the proton within Δt allowed by the uncertainty principle. Then the "virtual photon" has propagated from the electron to the proton, causing a scattering process, because of its kick when emitted by the electron and when absorbed by the proton. Since the photon is a massless particle with $E = cp$, its energy can be arbitrarily small for small momenta, and hence Δt can be arbitrarily long. The distance the "virtual photon" can propagate can also be arbitrarily long $d = c\Delta t$. This is why the electromagnetic interaction is long-ranged. If, on the other hand, the force carrier had a finite mass m, there is a minimum energy required to create the force carrier particle $E_{min} = mc^2$. Therefore the time to pay back the debt is limited: $\Delta t = \hbar/mc^2$. The distance the force carrier can go within the allowed time limit is then also limited: $d = c\Delta t = \hbar/mc$. Therefore the force carrier cannot go beyond this distance and the force becomes short-ranged. This distance determined by the mass of the particle is called "Compton wavelength." Yukawa suggested back in 30's that the force carrier of the nuclear force must therefore be massive. Judging from the range of the nuclear force of about two fm, he suggested that the force carrier must weigh about 200 times electron, or 100 MeV/c^2. The short-rangedness is then an immediate consequence of the finite mass.

The presence of the charge exchange reaction suggests that the force carrier is (or at least can be) electrically charged. This particle is called charged pion π^- or π^+ in the modern terminology. The charge exchange reaction, producing the backward peak in the *np* scattering is caused by the following process. When the neutron comes close to the proton, the neutron emits the force carrier π^-, and it becomes a proton (!). Even though (from the neutron point of view) she is still going pretty much straight ahead, we see the proton coming along the original direction of the neutron, namely the "backscattered proton." The emitted π^- is then absorbed within the time allowed by the uncertainty principle and the proton becomes a neutron.

By 40's there was discovered a particle that weighs 200 times electron in cosmic rays (or more precisely, 105.7 MeV$/c^2$). This of course raised hope that the discovered particle may be the force carrier for the nuclear force. After intensive research, however, especially that carried out by Italians hiding (literally) underground in Rome under Nazi's occupation in 1945, it was shown that the new particle does not show any sign to feel the nuclear force. This particle is what is now called muon $\mu^\pm$. Indeed, underground is a good place to study muons! Later on people speculated that there may be *two* new particles weighing 200 times electron, and this is indeed what happened. By going to higher altitudes on the Andes in cosmic ray studies, people have found that the charged pions exist in cosmic rays, which quickly (within about 10^{-8} sec) decay to muons which live longer (about 10^{-6} sec) and reach the surface of the Earth. (Of course their life is stretched by the relativisitc time dilation effect. Otherwise we didn't have a chance to detect them even on the Andes.) Only at higher altitudes, pions had chance to enter the detector (photographic films). Later on, a neutral pion π^0 was also discovered that decays into two photons. They are later determined to have no spin and odd parity. Once found, it seemed to confirm Yukawa's suggestion. The potential between nucleons caused by the exchange of a "virtual pion" was calculated to have the following form

$$V = \frac{1}{3}\frac{g^2}{\hbar c}\frac{m_\pi^2}{4m_N^2} m_\pi c^2 (\vec{\tau}_1 \cdot \vec{\tau}_2) \left[(\vec{\sigma}_1 \cdot \vec{\sigma}_2) + \left(1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2} \right) S_{12} \right] \frac{e^{-\mu r}}{\mu r}. \quad (6)$$

Here $\mu = m_\pi c/\hbar$ with m_π with the small difference between $m_{\pi^\pm} = 139.6$ MeV$/c^2$ and $m_{\pi^0} = 135.0$ MeV ignored in the same spirit as we ignore the proton-neutron mass difference and call it m_N. The factor

$$S_{12} = \frac{1}{r^2}[3(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r}) - (\vec{\sigma}_1 \cdot \vec{\sigma}_2)r^2] \tag{7}$$

is the form for the phenomenologically required tensor force. The matrices $\vec{\tau} = 2\vec{I}$ are the analogues of Pauli matrices for the isospin. The important point with the potential is that it is indeed invariant under the rotation of the isospin space because of the form $(\vec{\tau}_1 \cdot \vec{\tau}_2)$.

The OPE (one-pion-exchange) exchange Eq. (6) works well in the two-nucleon system. We have seen that there is only one bound state in two-nucleon system, namely deuteron, with $I = 0$, $L = 0$, $S = 1$. Let us see if this is consistent with the OPE potential. We focus on the s-wave ($L = 0$) which doesn't have the centrifugal barrier and presumably binds the most. When $I = 1$ ($(\vec{\tau}_1 \cdot \vec{\tau}_2) = +1$), the Fermi statistics requires $S = 0$ ($\vec{\sigma}_2 = -\vec{\sigma}_1$ and hence $(\vec{\sigma}_1 \cdot \vec{\sigma}_2) = -3$). Then the tensor force is proportional to

$$S_{12} = \frac{1}{r^2}[3(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r}) - (\vec{\sigma}_1 \cdot \vec{\sigma}_2)r^2] = \frac{1}{r^2}[-3(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_1 \cdot \vec{r}) + 3r^2]. \tag{8}$$

At the lowest order in the potential in perturbation theory, using the fact that the s-wave is isotropic, we find $\langle r^i r^j \rangle = \frac{1}{3}\langle r^2 \rangle$, and hence the tensor force vanishes identically. Therefore, the OPE potential is

$$V = -\frac{g^2}{\hbar c}\frac{m_\pi^2}{4m_N^2} m_\pi c^2 \frac{e^{-\mu r}}{\mu r}. \tag{9}$$

This potential is attractive, of finite range, and may or may not have a bound state depending on the size of the coupling $g^2/\hbar c$ and m_π. For the actual values, there is no bound state.

On the other hand, for the $I = 0$, $L = 0$, $S = 1$ case, we have $(\vec{\tau}_1 \cdot \vec{\tau}_2) = -3$ and $(\vec{\sigma}_1 \cdot \vec{\sigma}_2) = +1$. Let us take $S_z = +1$ state as an example. Then the tensor force does not vanish, and its expectation value is proportional to

$$\begin{aligned}
&\langle S = 1, S_z = +1| \frac{1}{r^2}[3(\vec{\sigma}_1 \cdot \vec{r})(\vec{\sigma}_2 \cdot \vec{r}) - (\vec{\sigma}_1 \cdot \vec{\sigma}_2)r^2]|S = 1, S_z = +1\rangle \\
&= \langle S = 1, S_z = +1| \frac{1}{r^2}[3(\sigma_1^z z)(\sigma_2^z z) - r^2]|S = 1, S_z = +1\rangle \\
&= \frac{2z^2 - x^2 - y^2}{r^2}.
\end{aligned} \tag{10}$$

Therefore, the OPE potential is

$$V = -\frac{g^2}{\hbar c}\frac{m_\pi^2}{4m_N^2} m_\pi c^2 \left[1 + \frac{2z^2 - x^2 - y^2}{r^2}\right] \frac{e^{-\mu r}}{\mu r}. \tag{11}$$

The coefficient of the potential is the same as the $I = 1$ case, except that there is an addition of the quadrupole moment $r^2 Y_2^0 = \sqrt{\frac{5}{16\pi}}(2z^2 - x^2 - y^2)$. If the quadrupole moment is positive, which means a cigar-like shape, as opposed to negative, which means a pancake like shape, the quadrupole moment adds to the attractive force and can lead to a bound state even if the $I = 1$ case doesn't. Experimentally, the quadrupole moment of the deuteron is confirmed and has the value $Q(d) = 2.78 \times 10^{-27}$ cm^2. The deuteron indeed has a cigar-like shape where the spins are lined up along the elongated direction.

5 Fundamental Description of Nuclear Force

Now the world looked simple: there are protons and neutrons in nuclei, bound together by the force mediated by the exchange of pions. But the world wasn't so simple after all. The first little problem is that the coupling needed for the pion-nucleon coupling was extremely big. The analogue of the fine-structure constant was

$$\frac{g^2}{\hbar c} \simeq 15. \tag{12}$$

Clearly, the perturbation theory which expands systematicaly in powers of $g^2 \hbar c$ is very badly behaved. Therefore the nuclei are very strongly coupled system and theoretically very hard to deal with.

The problem starts when you want to probe shorter distances. The first sign of the problem is the hard core in the nucleon-nucleon potential. The one-pion-exchange potential does not give you that. Then what about two-pion exchange? Remember the large coupling constant: the two-pion exchange is actually *bigger* than the one-pion exchange in general. Fortunately, the two-pion exchange would be suppressed beyond the distance $\hbar/(2m_\pi c)$, and hence the long-distance behavior is still valid with the one-pion exchange. But at shorter distances, more and more pion exchanges, or higher orders in $g^2/\hbar c$ are increasingly important and the perturbation theory is clearly not working. In general, this simple picture of the world starts faltering as you go to shorter distances, or equivalently, higher momentum transfers.[2]

The hell really broke loose when people discovered many more particles that participate in the nuclear force, collectively called *hadrons*. There are many more *mesons*, a general version of pions that are bosons, have integer spins. There are also many more *baryons*, a general version of nucleons that are fermions, have half-odd integer spins. These particles appear in the collision of nucleons and mesons as resonances. Are they all elementary particles? People believed for a long time that they are, because they are of the same family as protons and neutrons, which people firmly believed were elementary.

One organizing principle came out when people realized that the mass and the spin of hadrons of the same time (*i.e.*, same isospin, same parity, etc). By plotting the masses and spins on the so-called Chew–Frautschi plot[3] on the (m^2, J) plane, the hadrons of the same type fall on straight lines: $J = \alpha(0) + \alpha' m^2$. The intercept $\alpha(0)$ depends on the types, but α' came out more-or-less the same: $\alpha' \simeq (1.2\text{–}1.4\ \text{GeV}^{-2})$. This value led to the following picture: the hadrons are elastic strings, not particles. When the string is stretched, there is a constant tension T, giving the energy Tr where the r is the length of the string. This was the beginning of the string theory. If you regard Tr as a potential energy, a hand-waving analysis indeed gives the linear relation between $E^2 = (mc^2)^2$ and J. Write down the relativisic kinetic energy cp and the linear potential Tr:

$$H = cp + Tr. \qquad (13)$$

In the spirit of Bohr's argument, $pr = l\hbar$. Therefore,

$$E = \frac{\hbar c l}{r} + Tr. \qquad (14)$$

Minimizing it with respect to r, we find the average length of the string

[2] Another sign of the problem is that the magnetic moments of proton and neutron are anomalous: $g_p/2 = 2.79$ and $g_n/2 = -1.91$ as opposed to the Dirac's values: $g_p/2 = 1$, $g_n/2 = 0$. One can try to explain the numbers by the quantum fluctuation of pions in the vacuum, as we do in the Lecture Note "QED," but again the non-convergence of perturbation series makes it impossible to draw a reliable conclusion.

[3] This is Jeff Chew of our Department.

$r = \sqrt{\hbar c l/T}$ and its energy

$$E = \frac{\hbar c l}{\sqrt{\hbar c l/T}} + T\sqrt{\hbar c l/T} = 2\sqrt{\hbar c l T}, \tag{15}$$

and hence

$$m^2 = (E/c^2)^2 = \frac{4T}{c} l. \tag{16}$$

Indeed, the mass-squared is linear with the angular momentum l![4]

Even though this picture of elastic strings seems to reproduce the Chew–Frautschi relation qualitatively, what it meant was a (sort-of) return of Thomson-model of atoms: a jelly-like, fluffy, elastic, continuous medium rather than a hard-centered composite objects. This picture was proven to be false experimentally by an experiment at SLAC, bombarding protons by very energetic (in the standard of their time) electrons. This experiment is a repetition of the Rutherford experiment, called Deep Inelastic Scattering experiment, except that his α-particle is replaced by the electron (this is a good idea because the electron is truly elementary as far as we know and we don't need to worry about its structure to interpret the data) and the atom by the proton. Similarly to the Rutherford experiment, they saw electrons nearly backscattered: something impossible by an elastic string. What it means is that there are something hard and tiny inside the proton, which Feynman later called "partons." They indeed measured the form factor of the partons by studying the dependence of the cross section as a function of the momentum transfer, very similar to the discussion we had with the nuclear form factor. It turns out that the form factor is nearly independent of the momentum transfer, which implies that the "partons" are point-like, apparently behaving as free particles. (Remember the form factor is the Fourier

[4] The correct quantum mechanical treatment of a relativistic string turned out to be much more difficult. You start with Nambu–Goto action and quantize it, and find that the quantization procedure is consistent only in 26-dimensional spacetime. Even that case predicts a tachyon, a particle with a negative mass-squared, whose presence violates causality because it would go faster than the speed of light. A supersymmetric version happily gets away with tachyons, but still live in 10-dimensional spacetime. But the interesting thing about it was that it predicts a massless spin-two particle, which we don't see in the world of hadrons but can be identified with the graviton. Since then the string theory switched its gear from the would-be theory of hadrons to the "Theory of Everything," including quantum gravity.

transform of the charge distribution. A constant is the Fourier transform of the delta function.)

Even that didn't convince people that the proton was a composite object for a while. One of the main reason was that, in order to reproduce the observed pattern of hadrons in terms of point-like constituents, the "partons" had to have fractional electric charges, as pointed out by Gell-Mann and Neeman. Gell-Mann named them "quarks." The constituent of the nucleons and pions are supposed to be "up" and "down" quarks, with electric charges $+\frac{2}{3}|e|$ and $-\frac{1}{3}|e|$. Nobody (except a few wrong experiments) could find fractionally charged objects. Only in the second half of 70's, after the so-called November Revolution in Particle Physics when the teams at SLAC and Brookhaven independently discovered a particle now called J/ψ, people started to take the quark model seriously. The J/ψ is now understood as a boundstate of a charm quark and its anti-particle $c\bar{c}$, an entirely new type of quark not seen earlier. Still, people had to answer the question why fractionally charged quarks cannot be seen in isolation. The quarks must somehow be "confined" inside hadrons.

Now you can reinterpret the "tension" of the string as a potential between a quark and an anti-quark inside a meson. The potential Tr is linear in r, and hence there is no way for the quark to get isolated; it would cost an infinite amount of energy. It turns out that at some distance, it becomes energetically more favorable to create an additional pair of a quark and an anti-quark, so that the original meson splits into two mesons. No isolated quarks. By reinterpreting p in the analysis by the momentum of the quark, not the rotational motion of the string, you get qualitatively right spectrum. But then what is causing the linear potential between a quark and an anti-quark? And why can such a strong force somehow be neglected in the Deep Inelastic Scattering experiments where the partons behave as free particles? The answer is the Quantum ChromoDynamics, a theory of quarks and gluons. The gluon plays the role of the photon in Quantum ElectroDynamics. It turns out, however, that the gluon produces a linear potential between "charged" particles (because they are three different types of charges, they are usually called "colors" instead, even though they have nothing to do with optical spectrum). The precise mechanism behind the confinement is still under active research.

What, after all, was the Yukawa's theory of nuclear force by the pion exchange, then? It is now understood as a van der Walls like force among bound states. The van der Waals focrce acts among neutral atoms, which do not have any overall electric charges. But the residual effects due to the quantum polarizability make them attract each other. In the case of nucleon-nucleon potential, the situation is similar but somewhat different. A proton is a bound state of *uud* quarks while a neutron of *udd* quarks. Because they share the same constituents, you may interchange them. If a proton wants to interchange one of its quarks with the neutron, it needs to "send", say, an up-quark to the neutron. But before the up-quark reaches the neutron, it starts feeling the linear potential, and realizes that it needs to be bound with something: it then creates a pair of, say, a down-quark and an anti-down quark. The created down-quark stays with the rest of the proton: it is now a *dud* state and has become a neutron. The created anti-down quark $\bar{d}$ goes together with the "sent-out" up-quark forming a charged pion $u\bar{d}$. This charged pion can now propagate from what-used-to-be-a proton to the neutron. The $\bar{d}$ inside the charged pion annihilates together with one of the d quarks in the neutron, and the up-quark gets together with the rest of the neutron. It is now a *uud* state and has become a proton. This is the charge-exchange reaction via the one-pion exchange. It is still fine, except that it is only an effective description of what is truly going on useful only at relatively long distances (even though it is very short from the daily-life point of view).

6 Fermi Gas

We have learned that the nucleons interact strongly with each other. Obviously, the multi-body system of strongly interacting particles would be a very hard subject. But at least we should give it a try. And fortunately, as we discussed earlier already with the multi-electron atoms, Fermi gas (Thomas–Fermi) works quite well even when the interactions among particles are quite strong, because the anti-symmetry of the wave function takes care of the bulk of the effects of the interactions.

The starting point is the Fermi gas, stacking up nucleons up to the Fermi level $E_f = \vec{p}_F^2/2M$. Here, M is the nucleon mass, ignoring the difference between neutron and proton. The number density ρ of the nucleons has contributions from both neutrons and protons, each with two spin states. Therefore,

$$\rho = 4 \int^{p_F} \frac{d\vec{p}}{(2\pi\hbar)^3} = 4 \frac{4\pi}{3} p_F^3 \frac{1}{(2\pi\hbar)^3} = \frac{2}{3\pi^2\hbar^3} p_F^3. \tag{17}$$

Giving the more-or-less constant density of nuclei $\rho = 0.172$ nucleons/fm^3, we find $p_F = 268$ MeV/c or $k_F = p_F/\hbar = 1.36$ fm^{-1}. The corresponding Fermi energy is $E_F = 38$ MeV. The average kinetic energy of the nucleon is

$$\langle \frac{\vec{p}^2}{2M} \rangle = \frac{1}{\rho} 4 \int^{p_F} \frac{d\vec{p}}{(2\pi\hbar)^3} \frac{\vec{p}^2}{2M} = \frac{3}{5} E_F = 23 \text{ MeV}. \tag{18}$$

The empirical mass formula gave us the volume term to be 15.68 MeV per nucleon, and hence the potential energy must be the the sum of the binding energy and the kinetic energy, $\langle V \rangle \simeq -(16 + 23) = -39$ MeV. Neutron scattering on complex nuclei allows us to estimate the depth of the potential of around -40 MeV, in rough concordance with this naive estimate.

The symmetry term can be easily estimated in the Fermi gas model. Because we stack up neutrons and protons independently in the Fermi gas model, different $N \neq Z$ means that we use different Fermi energies for them. Clearly, the case with the same Fermi energies would give you the lowest total energy. That is a contribution to the symmetry term. Here, we don't expect a good agreement with data because the nature of the nucleon-nucleon interaction favors isosinglet, as we saw in the two-nucleon system, and hence the Fermi gas (without any interactions by definition) cannot account for the entire symmetry term. In any case, we can estimate it in the following way. We have N neutrons and Z protons in the same volume Ω. The Fermi momenta for neutrons and protons are, respectively, determined by the equations

$$N = \Omega \frac{1}{3\pi^2\hbar^3} p_{F,n}^3, \qquad Z = \Omega \frac{1}{3\pi^2\hbar^3} p_{F,p}^3. \tag{19}$$

Then the total energies given by the integral in Eq. (18) are

$$E_n = N \frac{3}{5} \frac{p_{F,n}^2}{2M} = \frac{3\hbar^2}{10M} \left(\frac{3\pi^2}{\Omega} \right)^{2/3} N^{5/3}, \qquad E_p = \frac{3\hbar^2}{10M} \left(\frac{3\pi^2}{\Omega} \right)^{2/3} Z^{5/3}. \tag{20}$$

Now we write down the total energy $E_{tot} = E_n + E_p$ using $Z = \frac{A}{2} - \frac{N-Z}{2}$, $N = \frac{A}{2} + \frac{N-Z}{2}$, $\rho = A/\Omega$, and expand in $N - Z$ to the second order and find

$$E_{tot} = \frac{3\hbar^2}{10M} \left(\frac{3\pi^2 \rho}{2} \right)^{2/3} A \left\{ 1 + \frac{5}{9} \left(\frac{N-Z}{A} \right)^2 + O\left(\frac{N-Z}{A} \right)^4 \right\}. \tag{21}$$

The first term is the kinetic energy contribution to the volume term we estimated earlier. The second term is the symmetry term. Using the constant density $\rho = A/\Omega = 0.172$ nucleons/fm^3,

$$E_{\rm sym} = \frac{3\hbar^2}{10M}\left(\frac{3\pi^2\rho}{2}\right)^{2/3}\frac{5}{9}\frac{(N-Z)^2}{A} = 12.8 \text{ MeV}\frac{(N-Z)^2}{A}. \qquad (22)$$

The dependence $(N-Z)^2/A$ is precisely what we need. However, this accounts for only about a half of the symmetry term $a_{\rm sym} = 28.1$ MeV. The other half must come from the increase in interaction energies when the number of protons and neutrons are not equal, *i.e.*, as the total isospin of the nucleus increases.

One can also try to estimate the surface term assuming a profile for the density as a function of the radius. I do not going into the discussion here.

The Coulomb term is estimated just by calculating the Coulomb energies among protons in the nucleus. Similarly to the calculations in the Hartree–Fock model of atoms, we need to calculate both the direct and exchange terms.

$$\begin{aligned} E_{\rm Coulomb} &= \frac{1}{2}22\sum_{\vec{k}_1,\vec{k}_2}[\langle\vec{k}_1,\vec{k}_2|\frac{e^2}{r_{12}}|\vec{k}_1,\vec{k}_2\rangle - \langle\vec{k}_1,\vec{k}_2|\frac{e^2}{r_{12}}|\vec{k}_2,\vec{k}_1\rangle] \\ &= 2\int\frac{d\vec{k}_1 d\vec{k}_2}{(2\pi)^6}d\vec{x}_1 d\vec{x}_2\frac{e^2}{r_{12}}\left[1-e^{-i(\vec{k}_1-\vec{k}_2)\cdot(\vec{x}_1-\vec{x}_2)}\right] \\ &= 2\int d\vec{x}_1 d\vec{x}_2\frac{e^2}{r_{12}}\left[\left(\frac{1}{6\pi^2}k_F^3\right)^2 - \left\{\frac{1}{2\pi^2}\frac{\sin k_F r_{12} - k_F r_{12}\cos k_F r_{12}}{r_{12}^3}\right\}^2\right] \end{aligned} \qquad (23)$$

The integrand vanishes when $k_F r_{12} \to 0$, signaling the "Fermi hole" we talked about in multi-electron atoms. The first term is just the Coulomb energy of a uniformly charged sphere,

$$2\int d\vec{x}_1 d\vec{x}_2\frac{e^2}{r_{12}}\left(\frac{1}{6\pi^2}k_F^3\right)^2 = \frac{3}{5}\frac{Z^2e^2}{R} = 0.77 \text{ MeV}\frac{Z^2}{A^{1/3}}, \qquad (24)$$

in good accordance with the coefficient 0.717. Note that we used $Z = 2(k_F^3/(6\pi^2))\Omega$ where the volume of the sphere is $\Omega = 4\pi R^3/3$. The second term cannot be integrated analytically inside a sphere $|\vec{x}_1|, |\vec{x}_2| < R$. Fortunately, the integrand asymptotes to zero practically beyond $k_F r_{12} \gtrsim 3$. If you assume that the radius of the nucleus is bigger ($k_F R \gg 1$), the integration over r_{12} can be done independent of the size of the nucleus, and we find

$$
\begin{aligned}
& 2\int d\vec{x}_1 d\vec{x}_2 \frac{e^2}{r_{12}} \left\{ \frac{1}{2\pi^2} \frac{\sin k_F r_{12} - k_F r_{12}\cos k_F r_{12}}{r_{12}^3} \right\}^2 \\
&= 2\Omega \int d\vec{x}_{12} \frac{e^2}{r_{12}} \left\{ \frac{1}{2\pi^2} \frac{\sin k_F r_{12} - k_F r_{12}\cos k_F r_{12}}{r_{12}^3} \right\}^2 \\
&= \frac{9\pi}{4} \frac{Z^2 e^2}{\Omega k_F^2} [1 - j_0(k_F R)^2 - j_1(k_F R)^2]. \qquad (25)
\end{aligned}
$$

For $k_F R \gg 1$, the spherical Bessel functions in the square bracket are negligible, and we find the total Coulomb energy to be

$$
E_{\text{Coulomb}} = 0.77 \text{ MeV} \frac{Z^2}{A^{1/3}} \left(1 - \frac{1.0}{A^{2/3}} \right). \qquad (26)
$$

Empirical fit to the data gives

$$
E_{\text{Coulomb}} = 0.717 \text{ MeV} \frac{Z^2}{A^{1/3}} \left(1 - \frac{1.69}{A^{2/3}} \right), \qquad (27)
$$

in reasonably good agreement with the naive Fermi gas model in a rigid sphere. The descrepancy in the exchange correction is attributed to the sharp cutoff we assumed at the radius R which needs to be smoother in reality.

7 Shell Model

Overall, it is interesting that the naive Fermi gas model works reasonably well even for a strongly interacting system like nuclei. But the only agreement is for the gross property such as the emperical mass formula. Once one asks more detailed questions, such as the spin-parity of the ground state, excitation spectrum, etc, we need more detailed models. We will stick to the independent-particle approximation, namely Fermi liquid, but prepare the single-particle wave function is a little bit more realistic manner. That is the shell model discussed in this section.

One important observation in the binding energies of nuclei is that there appears to be special numbers of nucleons for which the nucleus becomes particularly tightly bound. Such numbers are called the *magic numbers*. Look at the neutron-pair separation energies in Fig. 6. (It is better to look at the neutron-pair separation energies than a single neutron seration energies because of the pairing force.) The lines show the dependence of the separation energies for fixed N as a function of Z, which steadily increases consistent with the volume term of the empirical mass formula. The interesting comparison is among the lines. As you increase N, the separation energy decreases. But above $N = 82$ and 126, the decrease is dramatic while the amount of decrease is more-or-less the same above and below these gaps. This is a signal that the neutrons fill up a shell up to $N = 82$, beyond which they start filling the next shell which is less bound, and similarly for 126. A systematic study of this type showed that the magic numbers are

$$2, 8, 20, 28, 50, 82, 126 \tag{28}$$

The shell model is normally introduced in successive refinements in the following manner. We treat the nucleus with the mean-field potential which we hope approximates the inter-nucleon attractive force. We determine the energy levels with the mean-field potential, and fill nucleons into the energy levels independently. What potential shall we take? Because the nuclei have finite size and the nuclear force is short-ranged, the potential must also be short-ranged, practically zero outside the nucleus. Therefore a crude approximation would be a spherical well potential, but not with a step function at some radius. It needs to be an attractive potential well with relatively constant potential energy inside the nucleus, while smoothly vanishing outside the nucleus.

To avoid getting into numerical problems, we approximate the mean field potential by a harmonic oscillator potential initially. Then we "lower" the potential at large radius to take the vanishing of the potential outside the nucleus into account. Finally we introduce the spin-orbit coupling which is quite important. The last point was realized by Mayer and Haxel-Jensen-Suess in 1949, which led to the widely-used shell model of nuclei.

The three-dimensional isotropic harmonic oscillator has a very simple spectrum. We have three independent creation/annihilation operators a_x, a_y, a_z, with the same frequency because of the isotropy. The Hamiltonian is then

$$H = \hbar\omega(a_x^\dagger a_x + a_y^\dagger a_y + a_z^\dagger a_z) \tag{29}$$

where we ignored the zero-point energies. Because of the isotropy, we must be able to label states according to the angular momentum of the states. The ground state is obviously an s-state. Let us call it $1s$ state. The first excited

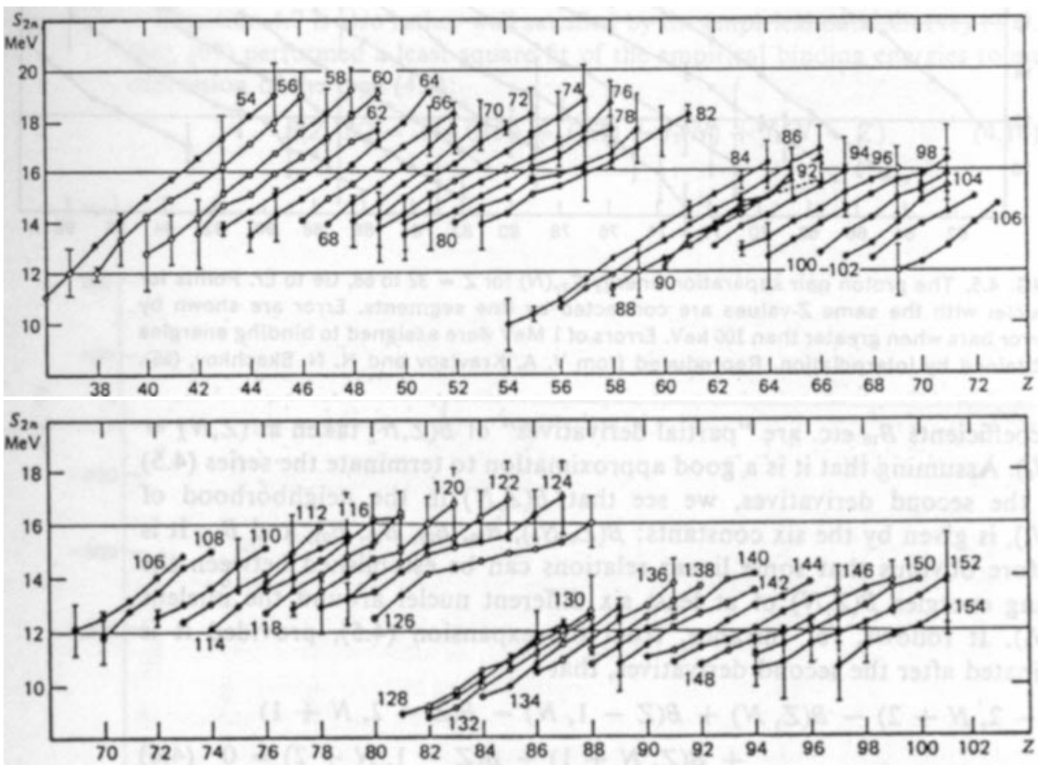

Figure 6: The neutron separation energy $S_{2n}(Z)$ for $N = 54$ to 154. Points for nuclei with the same N-values are connected by line sigments. N is indicated at the line. Errors are shown by error bars when greater than 100 keV. Errors of 1 Mev were assigned to binding energies obtained by interpolation. Reproduced from V.A. Kravtsov and N.N. Skachkov, 1966. Taken from "Theoretical Nuclear Physics," by Amos deShalit and Herman Feshbach, New York, Wiley, 1974.

state is obtained by acting one of the creation operators on the ground state, giving three states. Because (a_x, a_y, a_z) transforms as a vector under rotation, it is the same as the $Y_1^{1,0,-1}$, hence it is an p-state. Because of the historical reason, we call it $1p$ state rather than $2p$ state unlike in the hydrogen atom. The rule is that we call it $1x$ state whenever the x-wave appears for the first time, and $2x$ the second time, and so on. Also, (a_x, a_y, a_z) is odd under parity and hence the $1p$ state has odd parity. In general, states at even levels $E = 2n\hbar\omega$ have even parity, while those at odd levels $E = (2n-1)\hbar\omega$ odd parity. The second excited state has two creation operators. There are three states using the same creation operator twice, and three states using two different creation operators, giving six states in total. Combination $[(a_x^\dagger)^2 + (a_y^\dagger)^2 + (a_z^\dagger)^2]|0\rangle$ is invariant under rotation, and hence the $2s$-state, while the other five form multiplets of $1d$-state. Both of them have even parity. The third excited states are $2p$ and $1f$ with odd parity, the fourth excited state $3s$, $2d$, $1g$ with even parity, and so on. The harmonic oscillator labels are shown on the very left in Fig. 7. The would-be magic numbers with the harmonic oscillator potential are: 2, 8, 20, 40, 70, 112, 168. The first three agree with the observed magic numbers, while the latter three don't.

The shell model potential flattens beyond the radius of the nucleus, and hence brings the energy down at large radii. Because higher orbital angular momenta correspond to larger radii, the states with higher angular momenta come down. Therefore, within the second excited levels, the $1d$ states are lower than the $2s$ state. Simiarly, the $1f$ states are lower than the $2p$ states among the thrid excited levels. This ordering is shown on the 2nd column in Fig. 7. We then turn on the spin-orbit coupling $\vec{L}\cdot\vec{S}$. States with the orbital angular momentum l are split into states with $j = l + 1/2$ and $j = l - 1/2$. The shifts in the energy eigenvalues are proportional to $\vec{L}\cdot\vec{S} = (j(j+1) - l(l+1) - 3/4)/2$. Which one is higher depends on the sign of the spin-orbit coupling. We assume the opposite sign from that in the hydrogen atom to be consistent with the data. As a result, states with higher j are lower. The splitting is larger for larger l. The ordering of states shown in Fig. 7 is obtained by appropriately choosing the size of the spin-orbit coupling. Most importantly, the largest l states within a given harmonic oscillator level are at the bottom with flattened potential and are further brought down the most because of the spin-orbit coupling. They join the lower harmonic oscillator level and change the magic numbers. We now find the magic numbers to be; 2, 8, 20, 28, 50, 82, 126, 184. This set of magic numbers is in perfect agreement with the empirical one Eq. (28).

The shell model tells us more than the magic numbers. For instance, Fig. 8 shows the low-lying excitations of ^{90}Zr nucleus. ^{90}Zr has 40 protons and 50 neutrons. 50 Neutrons form a close shell, filling up to $1g_{9/2}$. 28 of 40 protons fill first four shells, while the remaining 12 fill $2p_{3/2}$, $1f_{5/2}$, and $2p_{1/2}$. If you excite one of the protons in $2p_{1/2}$ to $1g_{9/2}$, the remaining proton in $2p_{1/2}$ and the proton in $1g_{9/2}$ can form states with odd parity and $J = 4$ and 5. There are indeed 4^- and 5^- states. 5^- state is lower presumably because two protons are closer in space by lining up the orbital angular momenta. If you excite both protons from $2p_{1/2}$ to $1g_{9/2}$, it could give $J = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$; but the anti-symmetry of the wave function leaves only $J = 0, 2, 4, 6, 8$ as possibilities. They should all have even parity. Indeed, we see 0^+, 2^+, 4^+, 6^+, 8^+, in this order. The ordering is due to the pairing, which favors the state to be paired with its time-reversed one. This way, we can understand the level structure of nuclei when they are close to the closed-shell configurations. Fig. 8 also shows the calculated levels based on a simple model.

As you see, the shell model works quite well, not only explaining the magic numbers but also the level structures. It can also be used to calculate transition matrix elements in γ-decays (we will do this for atoms when we quantize the radiation field), magnetic moments and other multiple moments of nuclei, etc. However, I have to remind you that the shell model is based on the mean field potential, which has not been derived from the first principle, and is also based on the independent particle approximation. It for example ignores all correlation effects. We had learned in atomic physics that the independent particle approximation (Hartree–Fock model) works quite well. But there, the mean field potential could be *calculated*. In nuclear physics, the shell model potential is assumed based on empirical facts rather than calculated. One of the difficulties in calculating the mean field potential is that the nuclear force is not only two-body interaction (as in the Coulomb force), but also has multi-body potentials because it is not a fundamental interaction but rather a residual force.

8 Deformed Nuclei

If you go away from the closed-shell configurations, the level structure can become much more complex, with many levels closely located. In such nuclei,

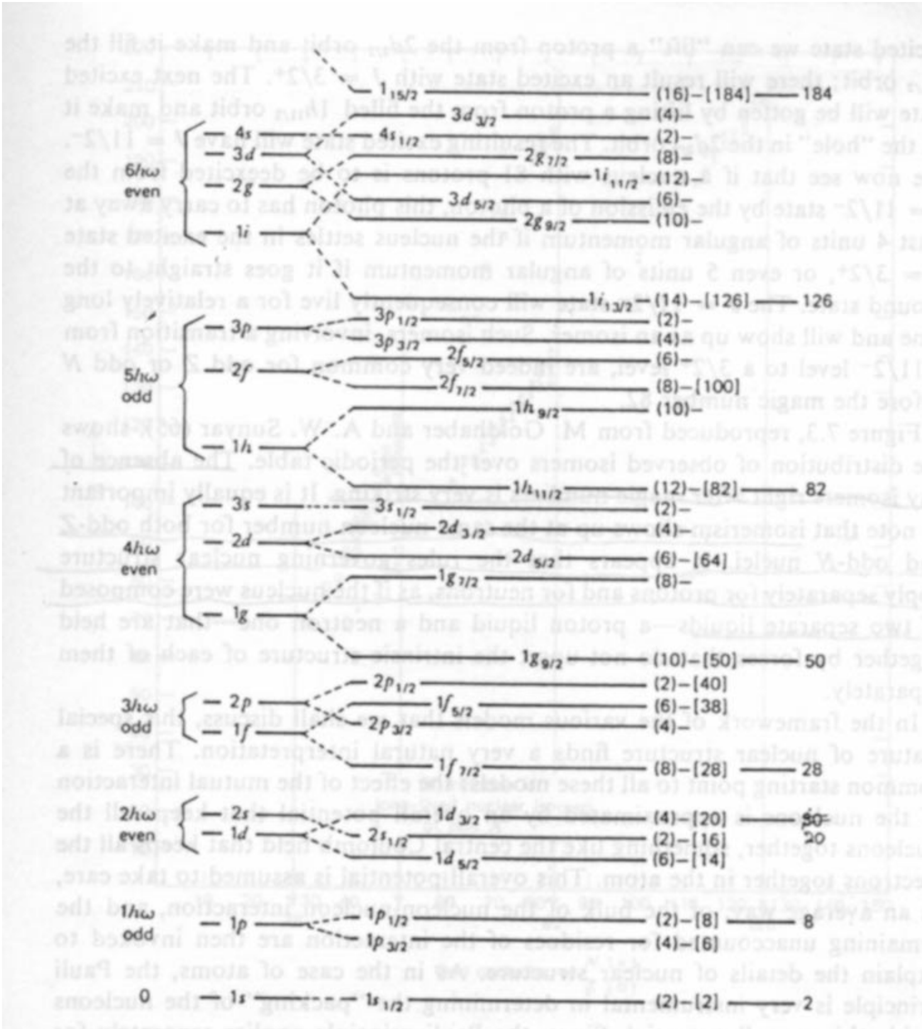

FIG. 7.2. Approximate level pattern for protons. The spin-orbit splitting is adjusted in such a way that the empirical level sequence is represented. For convenience the oscillator-level grouping and the parities of these groups are indicated at the left side of Fig. 7.2. Round brackets (2), (4), etc. and square brackets [2], [6], etc. indicate the level degeneracies and the total occupation numbers. In the 6ħω oscillator group the splittings are not drawn in a proper scale, the 3d splitting is too large. A more accurate drawing would have confused the picture too much. [Taken from Mayer and Jensen (55)].

Figure 7: Taken from "Theoretical Nuclear Physics," by Amos deShalit and Herman Feshbach, New York, Wiley, 1974.

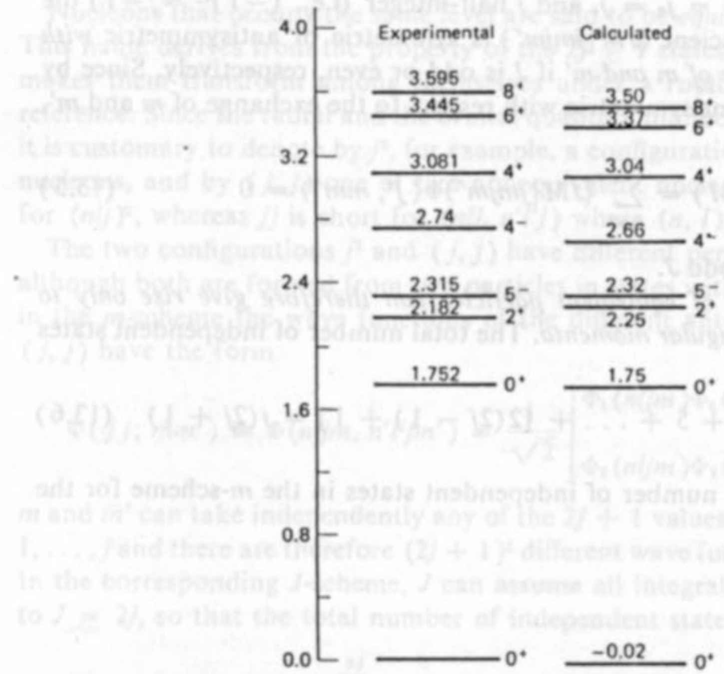

FIG. 14.1. Experimental and calculated energy levels of ^{90}Zr (in MeV). The total energies rather than energy differences, are plotted [taken from Auerbach and Talmi (65)].

Figure 8: Taken from "Theoretical Nuclear Physics," by Amos deShalit and Herman Feshbach, New York, Wiley, 1974.

the mixing among levels becomes important and the Fermi surface (*i.e.*, the boundary between filled and unfilled states) can be deformed significantly. In nuclei, it results in dramatic consequences: the nuclei are indeed deformed in their shapes.

In Fig. 9, energy levels of ^{176}Yb ($Z = 70$, $N = 106$), ^{178}Hf ($Z = 72$, $N = 106$), and ^{178}W ($Z = 74$, $N = 104$) are shown. The first important point is that the excitation energies are very low. In ^{90}Zr, the first excited level was 1.75 MeV above the ground state. On the other hand, the first excited level is at 0.082 MeV for ^{176}Yb. What it means is that the single-particle configurations are closely located and they highly mix. Another interesting point with these spectra is that the quantum numbers are ordered in a highly regular way: 0^+, 2^+, 4^+, all the way up to 14^+ in the case of ^{178}W. These are the typical examples of *rotational levels*. These nuclei have roughly half-filled shells, and many single particle states are closely located, get mixed, and conspire in such a way that the nuclei get elongated: cigar-like shape. They can rotate like a rigid rotator, producing the rotational levels. The energy levels can be fit quite well with the simple formula

$$E(J) = AJ(J+1) + BJ^2(J+1)^2. \tag{30}$$

The first term is that of a rigid rotator. The coefficients A and B are shown for each nucleus in Fig. 9.

Fig. 10 shows the quadrupole moments of nuclei as a function of odd number of nucleons. (Remember all even-even nuclei have 0^+ ground states and hence their quadrupole moments vanish identically even if they are deformed. On the other hand, measuring moments of excited and hence short-lived states is difficult. That is why odd-nucleon nuclei are useful here.) Close to the magic numbers, the quadrupole moments almost vanish, while in between the magic numbers, the nuclei show large quadrupole moments.

To understand nuclei with large deformations, we must deviate from the isotropic shell-model potential. The starting point is the anisotropic harmonic oscillator

$$H = \hbar\omega_1(a_x^\dagger a_x + a_y^\dagger a_y) + \hbar\omega_2 a_z^\dagger a_z. \tag{31}$$

Nuclei show a variety of other collective excitations beyond the overall rotation discussed in this section. For instace, ^{208}Pb is a doubly-closed shell nucleus, and hence is completely isotropic and tightly bound. Because the next excitation is quite high in the shell-model language, what it does is to

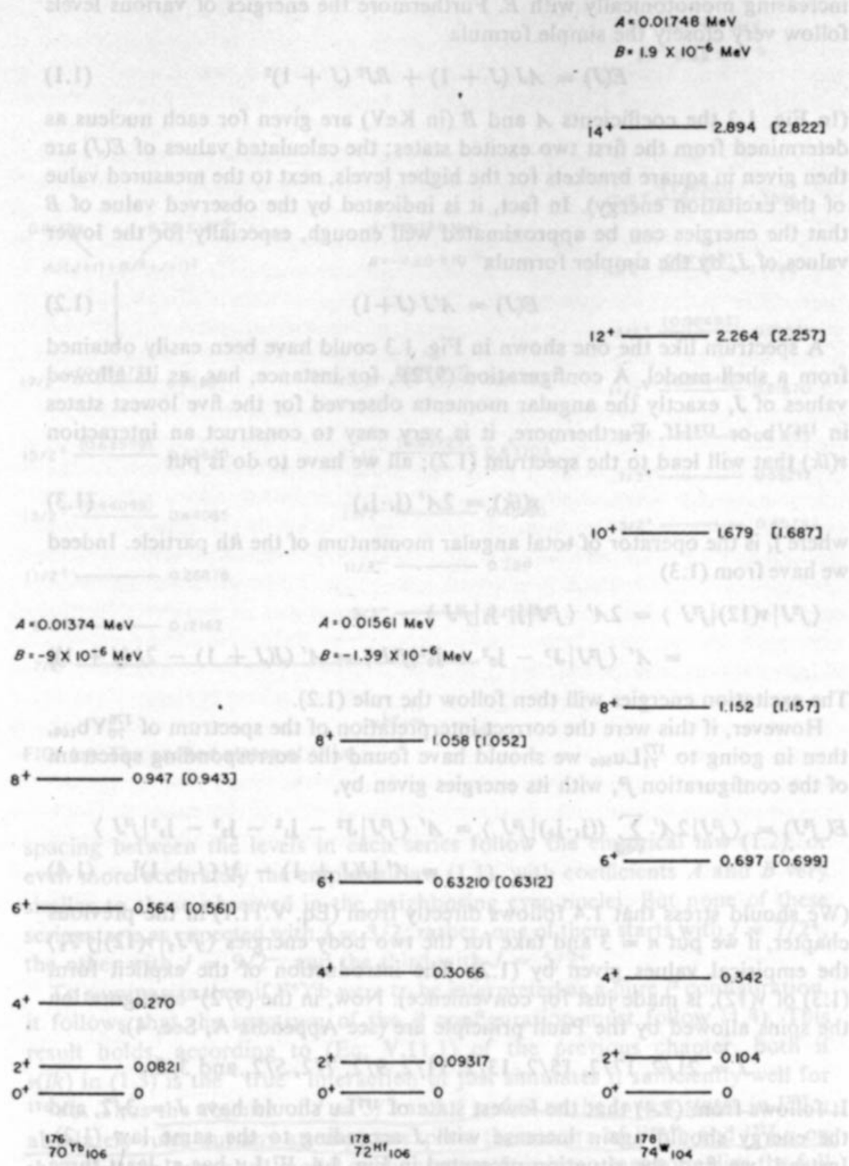

FIG. 1.3. Excited states of some deformed nucl… Energies are in MeV. The numbers in brackets give the energies of the expected states as calculated with (1.1) using indicated constants A and B in MeV.

Figure 9: Taken from "Theoretical Nuclear Physics," by Amos deShalit and Herman Feshbach, New York, Wiley, 1974.

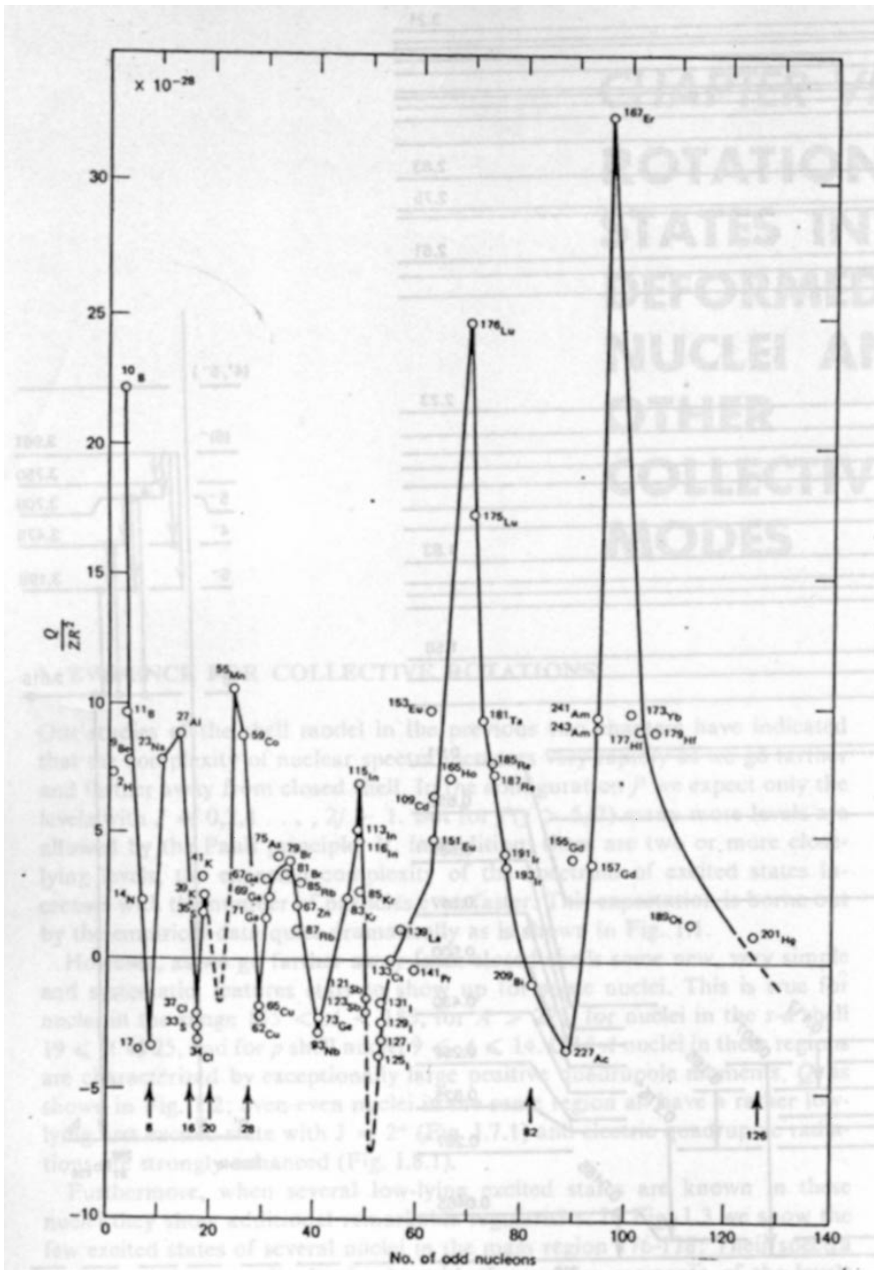

FIG. 1.2. Reduced nuclear quadrupole moments as a function of the number of odd nucleons. The quantity (Q/ZR^2) gives a measure of the nuclear deformation independent of the size of the nucleus [taken from Segre (64)].

Figure 10: Taken from "Theoretical Nuclear Physics," by Amos deShalit and Herman Feshbach, New York, Wiley, 1974.

create a small *octupole* deformation. The lowest excited state is 3^- and corresponds to a deformation to pear-like shape. There are quadrupole surface oscillations like those in a liquid drop.

9 Omissions

The discussion on nuclear physics here is very brief, focused only on static properties. I have not talked about dynamics, such as α-decay, scattering, fission, fusion, or β-decay.