HW #5 (221B), due Feb 23, 4pm

1. Consider nuclear α decay and calculate Gamov's transmission coefficients

$$T = \exp 2 \int_a^b \sqrt{\frac{2m(V(x) - E)}{\hbar^2}} dx, \tag{1}$$

where a, b are the classical turning points. Assume that the nucleus is a deep spherical well, *i.e.*, that a is the size of the nucleus. Consider following parameters: $Z = 92$ (Uranium), size of the nucleus 5 fm, and the kinetic energy of the α particle 1 MeV, 3 MeV, 10 MeV, 30 MeV.

2. Work out the WKB wave function for a harmonic oscillator. Compare the WKB wave function to the exact result given by Hermite polynomials for $n = 5$, and $n = 20$ by plotting them. Note that the WKB wave functions consist of classically forbidden parts, classically allowed parts, and Airy function interpolating between them.

3. The Maxwell equation (in Lorentz gauge) is

$$\left(\frac{n^2}{c^2} \frac{d^2}{dt^2} - \vec{\nabla}^2 \right) A^0(\vec{x}, t) = 0. \tag{2}$$

Consider it as a "Schrödinger equation" for the "light particle." Here, $n(\vec{x})$ is the index of refraction. Answer the following questions.

   (a) Writing $A^0 = e^{iS/\hbar}$, use "WKB Approximation" to write down the "Hamilton–Jacobi" equation for $S(\vec{x}, t)$.

   (b) Assume that the index of refraction depends only on x. Then we can separate variables t and y (forget z in this problem). Solve for $\tilde{S}(x, p_y, E) = S(x, y, t) - p_y y + Et$ as a function of x in an integral expression in the way you normally do for Hamilton–Jacobi equation.

   (c) Write down integral expressions for t and y.

   (d) Specialize to the case where $n(x) = n_1$ for $x < 0$ and $n(x) = n_2$ for $x > 0$. Show that the trajectory of the "light particle" follows the usual rule of refraction.

Note The geometrical optics is none other than the "WKB approximation" for the Maxwell's equation.