HW #8 (221B), due Apr 13, 4pm

1. Calculate the path integral with the imaginary time for the case of a simple harmonic oscillator. It is given by

$$Z = \int \mathcal{D}x(\tau) \exp\left[-\frac{1}{\hbar} \int_0^{\hbar\beta} d\tau \left(\frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\omega^2 x^2 \right) \right]. \tag{1}$$

   The integration variable $x(\tau)$ satisfies the periodic boundary condition $x(\hbar\beta) = x(0)$. You can Fourier expand it as

$$x(\tau) = \sum_{n=0}^{\infty} a_n \cos\frac{2\pi n\tau}{\hbar\beta} + \sum_{n=1}^{\infty} b_n \sin\frac{2\pi n\tau}{\hbar\beta}, \tag{2}$$

   and do the full path integral using the measure

$$\mathcal{D}x(\tau) = \prod_{n=0}^{\infty} da_n \prod_{n=1}^{\infty} db_n. \tag{3}$$

   Show that the result of the partition function is

$$Z = c\frac{e^{\beta\hbar\omega/2}}{e^{\beta\hbar\omega} - 1} \tag{4}$$

   where c is an overall constant that does not depend on ω.

2. Obtain eigenstates of the following Hamiltonian

$$H = \hbar\omega a^\dagger a + V a + V^* a^\dagger \tag{5}$$

   for a complex parameter V using the coherent states.

3. Show that the Bogliubov transformation

$$b = a\cosh\eta + a^\dagger \sinh\eta \tag{6}$$

$$b^\dagger = a^\dagger \cosh\eta + a\sinh\eta \tag{7}$$

   preserves the commutation relation of creation and annihilation operators $[b, b^\dagger] = 1$. Use this fact to obtain eigenvalues of the following Hamiltonian

$$H = \hbar\omega a^\dagger a + \frac{1}{2} V(aa + a^\dagger a^\dagger). \tag{8}$$

   Also show that the unitarity operator

$$U = e^{(aa - a^\dagger a^\dagger)\eta/2} \tag{9}$$

   can relate two set of operators $b = UaU^{-1}$.