HW #8 (221B)

1)

Path integrals with quadratic and lower polynomials in the exponent can be computed exactly, and this is the standard example. A book by Feynman and Hibbs is a good reference.

This path integral can be computed analagously to the one in the lecture notes "Quantum Field Theory II (Bose Systems)", the only significant difference being that in single-particle quantum mechanics, the degree of freedom $x(\tau)$ depends only on the (imaginary) time and not on space as do the fields $\psi(\vec{x}, \tau)$. Consequently, there are no $\vec{p}$ modes to integrate over.

When you use cosines and sines in the mode expansion, there will be a non-trivial Jacobian, which can be absorbed into the infinite constant. If you want to be more careful, one way to determine its value is to compute the path integral in the free-particle case and comparing the result with the known free-particle green's function.

Strictly speaking, you may find extra factors of β in your solution compared to Prof. Murayama's, which would seem to affect the thermodynamics. Recall that the original path integral is defined in terms of two integrals, $\int \mathcal{D}x(t)\mathcal{D}p(t)$, but we usually do the p integral implicitly en route to writing $Z = \int \mathcal{D}x(t)\, e^{-S/\hbar}$. If you go back and compute the p integral carefully, you will find just enough β dependence in your result to cancel the extra factors you find in doing the x integral.

2)

We want to "diagonalize" the Hamiltonian. When working with a and $a^\dagger$ operators this basically means trying to write it in the form $H \sim b^\dagger b$ where $[b, b^\dagger] = 1$. Staring at this Hamiltonian for a few minutes shows that it can be written

$$H = \hbar\omega(a^\dagger + \frac{V}{\hbar\omega})(a + \frac{V^*}{\hbar\omega}) - \frac{VV^*}{\hbar\omega},$$

and indeed $[b, b^\dagger] = 1$ for $b = a + \frac{V^*}{\hbar\omega}$ and $b^\dagger$ its complex conjugate. When b and $b^\dagger$ are adjoints and $[b, b^\dagger] = 1$, the states are determined uniquely to be what you expect:

$$|gs\rangle, \quad b^\dagger|gs\rangle, \quad \frac{1}{\sqrt{2!}}b^\dagger b^\dagger|gs\rangle, \quad \ldots, \tag{1}$$

where $b|gs\rangle = 0$. All we need to do is find the local ground state $|gs\rangle$. We know the coherent state $|f\rangle = e^{\frac{-ff^*}{2}} e^{fa^\dagger}$ is an eigenstate of the operator a

with eigenvalue f, so if we choose $f = -\frac{V^*}{\hbar\omega}$,

$$b\,|f = -\frac{V^*}{\hbar\omega}\rangle = (a + \frac{V^*}{\hbar\omega})\,|f = -\frac{V^*}{\hbar\omega}\rangle = 0.$$

So our ground state is the coherent state with $f = -\frac{V^*}{\hbar\omega}$, the eigenstates are given in (1), and the eigenvalues of the Hamiltonian are

$$E_n = \hbar\omega - \frac{VV^*}{\hbar\omega}, \qquad n = 0,\ 1,\ 2\ ,\ \ldots.$$

3)

Writing

$$b = a\cosh\eta + a^\dagger \sinh\eta, \qquad b^\dagger = a^\dagger\cosh\eta + a\sinh\eta,$$

$$[b, b^\dagger] = [a, a^\dagger]\cosh^2\eta + [a^\dagger, a]\sinh^2\eta = 1$$

when $[a, a^\dagger] = 1$.

Our given Hamiltonian is

$$H = \hbar\omega a^\dagger a + \frac{1}{2}V(aa + a^\dagger a^\dagger).$$

Adding another free parameter ξ and expanding in terms of a and $a^\dagger$,

$$\begin{aligned}\xi b^\dagger b &= \xi(a^\dagger a\cosh^2\eta + a^\dagger a^\dagger\cosh\eta\sinh\eta + aa\cosh\eta\sinh\eta + aa^\dagger\sinh^2\eta)\\ &= \xi(a^\dagger a\cosh 2\eta + (a^\dagger a^\dagger + aa)\tfrac{\sinh 2\eta}{2} + \sinh^2\eta),\end{aligned}$$

so to recover our Hamiltonian up to the constant $\xi\sinh^2\eta$, we require

$$\xi\sinh 2\eta = V, \qquad \xi\cosh 2\eta = \hbar\omega.$$

With a little bit of algebra you can find

$$\xi = \sqrt{(\hbar\omega)^2 - V^2}, \qquad \sinh^2\eta = \frac{1}{2}\left(\frac{\hbar\omega}{\sqrt{(\hbar\omega)^2 - V^2}} - 1\right),$$

and the Hamiltonian is

$$H = \sqrt{(\hbar\omega)^2 - V^2}\,b^\dagger b - \frac{1}{2}\hbar\omega + \frac{1}{2}\sqrt{(\hbar\omega)^2 - V^2}.$$

The eigenvalues can be read straight off.

As many of you noted this only works for $(\hbar\omega)^2 > V^2$, but anyway we usually think of V as a perturbation. This system almost corresponds to the lowest-order interacting Hamiltonian for a condensed bose gas as discussed in the lecture notes "Quantum Field Theory II (Bose Systems)". If we change our toy model Hamiltonian to

$$H \to \hbar\omega a^\dagger a + \frac{1}{2}V(aa + a^\dagger a^\dagger + 2a^\dagger a),$$

we have basically the zero-momentum version of eqn. (45) from the notes. Our diagonalized toy Hamiltonian becomes

$$H \to \sqrt{(\hbar\omega + V)^2 - V^2}\, b^\dagger b - \frac{1}{2}\hbar\omega + \frac{1}{2}\sqrt{(\hbar\omega + V)^2 - V^2},$$

which describes toy phonon excitations. Indeed $[b, b^\dagger] = 1$ implies we have bosons, and if we imagine $\hbar\omega \sim \frac{p^2}{2m}$, the small-$p$ expression for energy $\sqrt{(\hbar\omega + V)^2 - V^2}$ goes linear in p, the expected dispersion relation for sound compressions. The problem with $(\hbar\omega)^2 < V^2$ goes away.

For Hilbert space operators A, B, the Hausdorff formula reads

$$e^B A\, e^{-B} = A + [B, A] + \frac{1}{2!}[B, [B, A]] + \dots.$$

Barring mathematical niceties it is a clear consequence of Taylor expanding and rearranging, which makes me wonder why Hausdorff's (and sometimes Baker's or Campbell's) name is on it. To check that $b = UaU^{-1}$ for $U = e^{(aa - a^\dagger a^\dagger)\eta/2}$, use Hausdorff's formula to write

$$b = a + a^\dagger \eta + a\frac{1}{2!}\eta^2 + \dots = a\cosh\eta + a^\dagger \sinh\eta.$$