HW #9 Solutions (221B)

1)

This is a standard computation which can be found in most books on quantum field theory, though perhaps in the context of the scalar Klein-Gordon field.

$$H = \frac{1}{8\pi} \int d\vec{x}\ \vec{E}^2 + \vec{B}^2.$$

Using $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ and plugging in the mode expansion for $\vec{A}$, the $\vec{E}^2$ contribution to the energy is

$$\int d\vec{x}\ \vec{E}^2 \quad = \quad \int d\vec{x}\ \frac{1}{c^2}\frac{2\pi\hbar c^2}{L^3} \sum_{\vec{p},\vec{q},\lambda,\lambda'} (-i)^2 \sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}\ (\epsilon^i_\lambda(\vec{p}) a_\lambda(\vec{p}) e^{i\vec{p}\cdot\vec{x}/\hbar} - \epsilon^i_\lambda(\vec{p})^* a^\dagger_\lambda(\vec{p}) e^{-i\vec{p}\cdot\vec{x}/\hbar})$$

$$* \ (\epsilon^i_{\lambda'}(\vec{q}) a_{\lambda'}(\vec{q}) e^{i\vec{q}\cdot\vec{x}/\hbar} - \epsilon^i_{\lambda'}(\vec{q})^* a^\dagger_{\lambda'}(\vec{q}) e^{-i\vec{q}\cdot\vec{x}/\hbar}).$$

After multiplying out, rewrite

$$\int d\vec{x}\ e^{i(\vec{p}\pm\vec{q})\cdot\vec{x}/\hbar} \rightarrow (2\pi\hbar)^3 \delta^3(\vec{p}\pm\vec{q})$$

$$\sum_{\vec{q}} \rightarrow \frac{L^3}{(2\pi\hbar)^3} \int d\vec{q}.$$

Then since $\omega_{-\vec{p}} = \omega_{\vec{p}}$, after carrying out the obvious integrals we have

$$\int d\vec{x}\ \vec{E}^2 \quad = \quad -\sum_{\vec{p}} 2\pi\hbar\omega_{\vec{p}} \sum_{\lambda,\lambda'} (\epsilon^i_\lambda(\vec{p}) a_\lambda(\vec{p}) \epsilon^i_{\lambda'}(-\vec{p}) a_{\lambda'}(-\vec{p}) - \epsilon^i_\lambda(\vec{p})^* a^\dagger_\lambda(\vec{p}) \epsilon^i_{\lambda'}(\vec{p}) a_{\lambda'}(\vec{p})$$

$$- \epsilon^i_\lambda(\vec{p}) a_\lambda(\vec{p}) \epsilon^i_{\lambda'}(\vec{p})^* a^\dagger_{\lambda'}(\vec{p}) + \epsilon^i_\lambda(\vec{p})^* a^\dagger_\lambda(\vec{p}) \epsilon^i_{\lambda'}(-\vec{p})^* a^\dagger_{\lambda'}(-\vec{p})).$$

Now

$$\epsilon^i_\lambda(\vec{p}) \epsilon^i_{\lambda'}(-\vec{p}) = -\delta_{\lambda,\lambda'}$$

$$\epsilon^i_\lambda(\vec{p})^* \epsilon^i_{\lambda'}(\vec{p}) = \delta_{\lambda,\lambda'},$$

with analagous results for the other combinations (check simple cases). Then

$$\int d\vec{x}\ \vec{E}^2 \quad = \quad \sum_{\vec{p},\lambda} 2\pi\hbar\omega_{\vec{p}}\ (a_\lambda(\vec{p}) a_\lambda(-\vec{p}) + a^\dagger_\lambda(\vec{p}) a_\lambda(\vec{p}) + a_\lambda(\vec{p}) a^\dagger_\lambda(\vec{p}) + a^\dagger_\lambda(\vec{p}) a^\dagger_\lambda(-\vec{p})).$$

The terms like aa and $a^\dagger a^\dagger$ cancel with similar terms from $\vec{B}^2$ while the other terms add. Including the $1/8\pi$ from the definition of energy,

$$H = \frac{1}{8\pi} \int d\vec{x}\ \vec{E}^2 + \vec{B}^2 = \frac{1}{2} \sum_{\vec{p},\lambda} \hbar\omega_{\vec{p}}\ (a^\dagger_\lambda(\vec{p}) a_\lambda(\vec{p}) + a_\lambda(\vec{p}) a^\dagger_\lambda(\vec{p})).$$

Using $[a, a^\dagger] = 1$ gives the result

$$H = \sum_{\vec{p},\lambda} \hbar\omega_{\vec{p}}\, (a^\dagger_\lambda(\vec{p}) a_\lambda(\vec{p}) + \frac{1}{2}).$$

2)

We consider the coherent state of photons with $\vec{p} = (0, 0, p)$ and helicity $\lambda = +$.

$$|f, t\rangle := e^{-f^* f/2} e^{f e^{-ic|\vec{p}|t/\hbar} a^\dagger_+(\vec{p})} |0\rangle.$$

$$i\hbar \frac{\partial}{\partial t} |f, t\rangle = c\,|\vec{p}|\, f e^{-ic|\vec{p}|t/\hbar} a^\dagger_+(\vec{p}) |f, t\rangle.$$

Since $|f, t\rangle$ is an eigenstate of the annihilation operator, $a_\lambda(\vec{q})|f, t\rangle = \delta_{\lambda+}\delta_{\vec{p}\vec{q}}\, f e^{-ic|\vec{p}|t/\hbar}|f, t\rangle$,

$$H\,|f, t\rangle = \sum_{\vec{q},\lambda} c\,|\vec{q}|\; a^\dagger_\lambda(\vec{q}) a_\lambda(\vec{q})|f, t\rangle = c\,|\vec{p}|\; a^\dagger_\lambda(\vec{p})\; f e^{-ic|\vec{p}|t/\hbar}|f, t\rangle,$$

ignoring the zero point energy and using the delta functions to perform the sums. Clearly $i\hbar\frac{\partial}{\partial t}|f, t\rangle = H\,|f, t\rangle$. For another computation see p. 7 of the notes on the quantized radiation field.

Again, $|f, t\rangle$ is an eigenstate of the annihilation operator and $\langle f, t|$ is an eigenstate of the creation operator so that

$$\langle f, t|\, a_\lambda(\vec{q})|f, t\rangle = \delta_{\lambda+}\delta_{\vec{p}\vec{q}}\, f e^{-ic|\vec{p}|t/\hbar},$$

$$\langle f, t|\, a^\dagger_\lambda(\vec{q})|f, t\rangle = \delta_{\lambda+}\delta_{\vec{p}\vec{q}}\, f^* e^{ic|\vec{p}|t/\hbar}.$$

The definition of $\vec{A}$ gives immediately

$$\langle f, t|\vec{A}|f, t\rangle = \sqrt{\frac{2\pi\hbar c^2}{L^3}} \frac{1}{\sqrt{\omega_{\vec{p}}}} (\vec{\varepsilon}_+(\vec{p}) f e^{-ip\cdot x/\hbar} + \vec{\varepsilon}^{\,*}_+(\vec{p}) f^* e^{ip\cdot x/\hbar}),$$

where $p \cdot x = c\,|\vec{p}|\,t - \vec{p}\cdot\vec{x}$ is the Minkowski scalar product.

The coherent state expectation value reproduces a classical plane wave. We can learn more about the quantum–classical correspondance by considering the energy in this wave:

Professor Murayama discussed how the probability distribution in particle number behaves for a coherent state: It is a Poisson distribution with mean $\bar{n} = f^* f$. So set $f = \bar{n} e^{i\phi}$ where ϕ is some phase. Then with $\vec{\varepsilon}_+ = \frac{1}{\sqrt{2}}(1, i, 0)$,

$$\begin{aligned} \langle \vec{A} \rangle &= \sqrt{\frac{\pi \hbar c^2 \bar{n}}{L^3 \omega_{\vec{p}}}} \{ (e^{-ip\cdot x/\hbar + i\phi} + e^{ip\cdot x/\hbar - i\phi}) \hat{x} + (i e^{-ip\cdot x/\hbar + i\phi} - i e^{ip\cdot x/\hbar - i\phi}) \hat{y} \} \\ &= \sqrt{\frac{4\pi \hbar c^2 \bar{n}}{L^3 \omega}} \{ \hat{x} \cos(\vec{k} \cdot \vec{x} - \omega t + i\phi) - \hat{y} \sin(\vec{k} \cdot \vec{x} - \omega t + i\phi) \}, \end{aligned}$$

where $\vec{k} = \vec{p}/\hbar$ and $\omega = \omega_{\vec{p}} = c\,|\vec{p}|$. Now $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$, and the energy density U is $U = \frac{1}{4\pi} |\langle \vec{E} \rangle|^2$, which becomes

$$U = \frac{1}{4\pi} |\vec{E}|^2 = \frac{1}{L^3} \hbar \omega \bar{n}.$$

I was very excited the first time I saw this in Prof. Commins's class because it explains the transition between the frequency-based energy ($\hbar\omega$) of quantum mechanics and the amplitude-based energy ($|E|^2$) of classical electricity and magnetism in exactly the way you'd hope. Each photon contributes $\hbar\omega$ to the energy, and there are $\bar{n}$ photons on average in the state. The amplitude of the coherent state (which behaves like a classical wave) goes as $\sqrt{\hbar\omega\bar{n}}$.

3)

The Hamiltonian $H = -J \sum \vec{s}_i \cdot \vec{s}_j$, or its close cousin $H = -J \sum s_{zi} s_{zj}$ (called the Ising model), is the starting point for studies of phase transitions and critical phenomena in statistical mechanics. In the case of the ferromagnet, the Hamiltonian derives from the effective spin-spin interaction mediated by the Pauli principle, and not from the dipole-dipole interaction of the electrons, which is sufficiently weak that I can't find it discussed in any quantum mechanics books. Given the size of corrections from the Dirac treatment (e.g. spin-orbit) and hyperfine effects, I'm guessing the actual electron dipole-dipole interaction is at least 10^4 times weaker than the effective spin-spin interaction.

Consider a two-electron subsystem. Working in the approximation of single-particle wavefunctions, antisymmetry requires that

$$\psi(1,2) = (u_1(1) u_2(2) \pm u_1(2) u_2(1)) \otimes \begin{pmatrix} s = 0 \\ s = 1 \end{pmatrix},$$

where u_i are the spatial parts of the one-particle eigenfunctions and the antisymmetric ($s = 0$) spin wavefunction is paired with the symmetric spatial combination while symmetric ($s = 1$) spin go with antisymemtric spatial pieces. The Coulomb interaction between the two electrons contributes to the energy,

$$\langle\psi|\frac{1}{r_{12}}|\psi\rangle = \frac{K}{2} \pm \frac{J}{2},$$

where

$$\begin{array}{l} K = 4\langle u_1(1)u_2(2)|\frac{1}{r_{12}}|u_1(1)u_2(2)\rangle \\ J = 4\langle u_1(1)u_2(2)|\frac{1}{r_{12}}|u_1(2)u_2(1)\rangle. \end{array}$$

Explicitly,

$$\begin{array}{l} \langle\frac{1}{r_{12}}\rangle_{s=0} = \frac{K}{2} + \frac{J}{2} \\ \langle\frac{1}{r_{12}}\rangle_{s=1} = \frac{K}{2} - \frac{J}{2}. \end{array}$$

In this model of the ferromagnet, we assume the spins are confined to lattice sites but still require antisymmetry of the wavefunction. Then the energy between any two nearest neighbor pairs can be given solely in terms of the common integrals J, K,

$$H = \frac{K}{2} - J(\vec{s}_1 \cdot \vec{s}_2) - \frac{J}{4}.$$

Indeed, since

$$\vec{s}_1 \cdot \vec{s}_2 = \frac{1}{2}(s^2 - s_1^2 - s_2^2) = \frac{1}{2}(s^2 - 3/2),$$

the Hamiltonian takes values $\frac{K}{2} \pm \frac{J}{2}$ for $s = 0$ and $s = 1$ ($s^2 = 1(1+1) = 2$). Dropping the constants $K/2 - J/4$ and summing over all nearest neighbor pairs, we recover the Hamiltonian given in the problem set. The difference between our Hamiltonian and the Ising Hamiltonian is that in our case, there are three $s = 1$ states but only one $s = 0$ state, so three lower- and one higher-energy eigenvalues for H. In contrast, the Ising Hamiltonian $\sim \sum s_{z1}s_{z2}$ gives two states of lower energy and two states of higher energy. It neglects the x and y components of the spins.

In this problem we are asked to consider the ferromagnet ground states. As temperature decreases, the ferromagnet undergoes a second-order phase transition from disordered to ordered state where the bulk of the spins are aligned. The system undergoes "spontaneous symmetry breaking" in that the aligned spins "choose" to point in one of many equivalent directions. The ground state configuration breaks the original rotational symmetry of the problem. This is entirely analogous to how a Bose condensate "chooses" a particular phase θ for the field ψ, $\langle\psi\rangle = \sqrt{\rho}e^{i\theta}$, when it condenses. We discussed in class how this choice of phase for the ground state breaks gauge invariance when the ψ field describes Cooper pairs coupled to an E&M field in a superconductor. The breaking of gauge invariance makes the photon act as if it has a mass m_γ: magnetic fields are expelled from a superconductor up to exponentially dying penetration which falls off with distance r as $e^{-m_\gamma cr/\hbar}$.

a)

Work in units where $\hbar = 1$. When all spins are up along the z axis,

$$\vec{s}_i \cdot \vec{s}_j|\uparrow\uparrow\rangle = s_{xi}s_{xj} + s_{y_i}s_{y_j} + s_{zi}s_{zj}|\uparrow\uparrow\rangle = \frac{1}{4}(|\downarrow\downarrow\rangle - |\downarrow\downarrow\rangle + |\uparrow\uparrow\rangle) = \frac{1}{4}|\uparrow\uparrow\rangle.$$

Therefore

$$H|0\rangle = -J\sum_{\langle i,j\rangle} \vec{s}_i \cdot \vec{s}_j|0\rangle = -J\sum_{\langle i,j\rangle}\frac{1}{4}|0\rangle = E_0|0\rangle.$$

b)

The system is rotationally invariant, so the Hamiltonian should commute with the rotation operator. We can check this for the particular rotation $\widetilde{U} = \Pi_i U(\theta) = e^{-i\theta\sum_i s_{y_i}}$:

$$\begin{aligned}[s_{y_i} + s_{y_j}, \vec{s}_i \cdot \vec{s}_j] &= [s_{y_i} + s_{y_j}, s_{xi}s_{xj} + s_{y_i}s_{y_j} + s_{zi}s_{zj}] \\ &= -is_{zi}s_{xj} - is_{xi}s_{zj} + is_{xi}s_{zj} + is_{zi}s_{xj} = 0.\end{aligned}$$

Commuting operators have commuting exponentials, so $H\widetilde{U} = \widetilde{U}H$; the Hamiltonian is invariant under the rotation. This means that the new ground state $|0'\rangle := \widetilde{U}|0\rangle$ satisfies

$$H\widetilde{U}|0\rangle = \widetilde{U}H|0\rangle = E_0\widetilde{U}|0\rangle,$$

so the rotated state is also a ground state, an equivalent "choice" for the spontaneous symmetry breaking.

We want to check that the two ground states are orthogonal in the limit $N \to \infty$ where N is the number of spins. Consider a given spin which in the ground state is in the state $|\uparrow\rangle = \binom{1}{0}$. The rotation sends this to

$$|\uparrow'\rangle = \begin{pmatrix} \cos\frac{\theta}{2} - \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} + \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle.$$

Taking the inner product $\langle 0|0'\rangle$ will give a product of factors $\langle\uparrow|\uparrow'\rangle$, one for each spin. The factors are

$$\langle\uparrow|\uparrow'\rangle = \langle\uparrow|(\cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle) = \cos\frac{\theta}{2}.$$

For N spins,

$$\langle 0|0'\rangle = (\cos\frac{\theta}{2})^N.$$

For any non-zero rotation, the factor $\cos\frac{\theta}{2}$ will be less than one. Thus as $N \to \infty$, $(\cos\frac{\theta}{2})^N \to 0$.

c)

Let $|m\rangle = \Pi_i U(\theta_i)|0\rangle$. We want to compute $\langle m|H|m\rangle$ to $O(\theta^2)$:

$$\begin{aligned}\langle m|H|m\rangle &= \langle 0|(1 + i\textstyle\sum_i \theta_i s_{y_i} - \frac{1}{2}\sum_{ij}\theta_i s_{y_i}\theta_j s_{y_j} + \ldots) * H * \\ &\quad (1 - i\textstyle\sum_i \theta_i s_{y_i} - \frac{1}{2}\sum_{ij}\theta_i s_{y_i}\theta_j s_{y_j} + \ldots)|0\rangle.\end{aligned}$$

$O(0)$ terms give the ground state energy, and $O(\theta)$ terms vanish because $|0\rangle$ is an energy eigenstate and $\langle 0|s_{y_i}|0\rangle = 0$. The $O(\theta^2)$ terms are

$$-\frac{1}{2}\langle 0|\sum_{ij}\theta_i s_{y_i}\theta_j s_{y_j} H|0\rangle + \langle 0|\sum_i \theta_i s_{y_i} H \sum_j \theta_j s_{y_j}|0\rangle - \frac{1}{2}\langle 0|H\sum_{ij}\theta_i s_{y_i}\theta_j s_{y_j}|0\rangle$$

$$\begin{aligned}&= -\tfrac{1}{2}E\langle 0|\textstyle\sum_{ij}\theta_i s_{y_i}\theta_j s_{y_j}|0\rangle \quad + \quad \langle 0|\textstyle\sum_{ij}\theta_i s_{y_i}\theta_j s_{y_j} H|0\rangle \\ &\quad + \quad \langle 0|\textstyle\sum_i \theta_i s_{y_i}[H, \sum_j \theta_j s_{y_j}]|0\rangle - \tfrac{1}{2}E\langle 0|\textstyle\sum_{ij}\theta_i s_{y_i}\theta_j s_{y_j}|0\rangle\end{aligned}$$

$$= \langle 0|\sum_i \theta_i s_{y_i}[H, \sum_j \theta_j s_{y_j}]|0\rangle$$

$$= \frac{J}{8}\sum_{\langle i,j\rangle}(\theta_i - \theta_j)^2,$$

where the last sum runs over nearest-neighbor pairs. Therefore,

$$\langle m|H|m\rangle = E_0 + \frac{J}{8}\sum_{\langle i,j\rangle}(\theta_i - \theta_j)^2 + O(\theta^3).$$