Final Exam (221B), due May 11, 4pm

1. Consider the decay of the $2p$ state of hydrogen atom to the $1s$ state. Calculate the amplitude of the decay for $m = +1$ state using plane waves for photons, and explain the θ dependence of the amplitude for each helicity ± 1 of the final-state photon in terms of the angular momentum conservation. Show that the rate is the same as the decay rate of the $m = 0$ state.

2. How can the $2s$ state decay to the $1s$ state? Do not calculate the rate, but discuss it.

3. The coupling of the magnetic moment to the magnetic field $V = -\vec{\mu} \cdot \vec{B}$ can also cause transitions. (One such example is the hyperfine transition in hydrogen atom.) By expanding the Hamiltonian in multipoles, show that emission or absorption of a photon can change the spin state by M1 transitions.

4. The relativistic field equation for a spinless particle in the presence of the Maxwell field is

$$\left[\left(-i\hbar \frac{1}{c} \frac{\partial}{\partial t} - \frac{e}{c} A^0 \right)^2 - \left(-i\hbar \vec{\nabla} - \frac{e}{c} \vec{A} \right)^2 - m^2 c^2 \right] \phi = 0. \tag{1}$$

Answer the following questions.

(a) We would like to determine energy eigenvalue E in the presence of Coulomb potential $eA^0 = \frac{Ze^2}{r}$. Show that time-independent field equation for the radial wave function $\phi = R(r) Y_l^m e^{-iEt/\hbar}$ has the form

$$\left[\frac{\hbar^2}{2\mu} \left(-\frac{d^2}{dr^2} - \frac{2}{r}\frac{d}{dr} + \frac{\lambda(\lambda+1)}{r^2} \right) - \frac{Ze^2}{r} \right] R = \epsilon R. \tag{2}$$

Write μ, λ, ϵ in terms of E, m, and l.

(b) Eq. (2) has exactly the same form as the Schrödinger equation for the hydrogen atom, except that λ is not an integer. Therefore the bound-state eigenvalues are given by

$$\epsilon = -\frac{1}{2} \frac{Z^2 \alpha^2 \mu c^2}{\nu^2},$$

where the "principal quantum number" ν takes values $\nu = \lambda, \lambda + 1, \lambda + 2, \cdots$. Solve for E.

(c) Expand E up to $O(Z^2\alpha^2)$ and show that it agrees with the result of conventional Schrödinger equation including the rest energy.

(d) Expand E up to $O(Z^4\alpha^4)$, and discuss the interpretation of the correction.