HW #10 (221B), due Apr 26, 5pm

1. Consider a coherent state of photons in a particular momentum $\vec{p} = (0, 0, p)$ and helicity $+1$

$$|f\rangle = e^{-f^* f/2} e^{f a_+^\dagger(\vec{p})} |0\rangle. \tag{1}$$

Use the mode expansion of the Maxwell field,

$$A^i(\vec{x}) = \sqrt{\frac{2\pi\hbar c^2}{L^3}} \sum_{\vec{p}} \frac{1}{\sqrt{\omega_p}} \sum_{\pm} (\epsilon^i_\pm(\vec{p}) a_\pm(\vec{p}) e^{i\vec{p}\cdot\vec{x}/\hbar} + \epsilon^i_\pm(\vec{p})^* a^\dagger_\pm(\vec{p}) e^{-i\vec{p}\cdot\vec{x}/\hbar}) \tag{2}$$

$$\dot{A}^i(\vec{x}) = \sqrt{\frac{2\pi\hbar c^2}{L^3}} \sum_{\vec{p}} (-i\sqrt{\omega_p}) \sum_{\pm} (\epsilon^i_\pm(\vec{p}) a_\pm(\vec{p}) e^{i\vec{p}\cdot\vec{x}/\hbar} - \epsilon^i_\pm(\vec{p})^* a^\dagger_\pm(\vec{p}) e^{-i\vec{p}\cdot\vec{x}/\hbar}). \tag{3}$$

(1) Show that the Hamiltonian of photons is

$$H = \int d\vec{x} \frac{1}{8\pi} \left(\vec{E}^2 + \vec{B}^2 \right) = \sum_{\vec{p}} \sum_{\lambda} c|\vec{p}| \left(a^\dagger_\lambda(\vec{p}) a_\lambda(\vec{p}) + \frac{1}{2} \right). \tag{4}$$

Ignore the zero-point energy term below. (2) Show that the Schrödinger equation $i\hbar\frac{\partial}{\partial t}|f\rangle = H|f\rangle$ has a solution $|f, t\rangle = |fc^{-ic|\vec{p}|t/\hbar}\rangle$. (3) Calculate the expectation value of the Maxwell field $\langle f, t|\vec{A}(\vec{x})|f, t\rangle$. You can see that this state describes a classical electromagnetic wave such as laser.

2. Consider spins 1/2 on a lattice with nearest neighbor interaction for a ferromagnet

$$H = -J \sum_{\langle i,j \rangle} \vec{s}_i \cdot \vec{s}_j. \tag{5}$$

Here, i, j refer to lattice sites and $\langle i, j\rangle$ are nearest neighbor pairs. For simplicity, consider infinite number of spins lined up in only one dimension. Answer the following questions. You can use the identity

$$U(\theta) = e^{-i\theta \vec{s}_y} = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}. \tag{6}$$

(a) Show that the state with all spins up is an eigenstate of the Hamiltonian (it is actually the ground state).

(b) Show that $U(\theta)$ acting on all spins at the same time gives you another ground state which is orthogonal to the previous one in the limit of infinite number of spins.

(c) Consider a state

$$|k\rangle = \sum_n e^{ikna} |\cdots \uparrow \cdots \underset{n-1}{\uparrow} \underset{n}{\downarrow} \underset{n+1}{\uparrow} \uparrow \cdots \uparrow \cdots\rangle, \tag{7}$$

where a is the lattice constant. Show that this is an eigenstate of the Hamiltonian Eq. (5). Obtain the excitation energy as a function of k.