HW #3 (221B), due Feb 15, 4pm

1. Consider the "Delta-Shell" potential ($\gamma > 0$)

$$V(r) = \gamma\delta(r-a) \tag{1}$$

and the scattering problem for the S-wave. Answer the following questions.

   (a) In the limit $\gamma \to \infty$, the regions inside ($r < a$) and outside ($r > a$) the shell decouple. What are the values of k for the "bound states" confined inside the shell?

   (b) Show that the phase shift is given by

$$e^{2i\delta_0} = \frac{1 + \frac{2m\gamma}{\hbar^2 k} e^{-ika} \sin ka}{1 + \frac{2m\gamma}{\hbar^2 k} e^{ika} \sin ka} = e^{-2ika} \frac{\sin ka + \frac{\hbar^2 k}{2m\gamma} e^{ika}}{\sin ka + \frac{\hbar^2 k}{2m\gamma} e^{-ika}}. \tag{2}$$

   (c) Verify that $\gamma \to \infty$ gives the hard sphere, while $\gamma \to 0$ no scattering.

   (d) Plot the behavior of the partial wave cross section σ_0 for approriate values of parameters. Identify peaks due to the hard sphere scattering as well as resonances.

   (e) Identify the location of poles in the large γ approximation up to $O(\gamma^{-2})$, and see that the real values of k correpond to those of the "bound states" in the large γ limit.

   (f) Work out the wave function for the complex values of k at the poles analytically (no expansion in γ), and plot its time-dependence to identify the wave flowing to infinity resulting from the decay of the quasi-bound state.

   (g) Discuss the behavior of a wave packet whose momentum is peaked around the resonance value with a width much wider than the width of the resonance.

2. Using WKB approximation, answer the following questions.

   (a) Show that the phase shift is given by

$$\delta_l = \lim_{R\to\infty} \left[\int_{r'}^{R} \sqrt{k^2 - U(r) - \frac{l(l+1)}{r^2}} dr - \int_{r''}^{R} \sqrt{k^2 - \frac{l(l+1)}{r^2}} dr \right]. \tag{3}$$

   Here, r', r'' are the largest values of r where the respective argument of the square root becomes negative, and $U(r) = 2mV(r)/\hbar^2$.

   (b) Apply this approximation to the hard sphere scattering and compare it to the exact calculations. Due to some reason, Mathematica does not give useful integrals for this purpose. Change the integration variable by hand, and use Mathematica only afterwards to integrate and expand if needed.