HW #8 (221B), due Apr 12, 5pm

1. Calculate the path integral with the imaginary time for a boson. We focus only on one momentum mode $\psi(\vec{x}, \tau) = \frac{1}{L^{3/2}}\psi(\tau)e^{i\vec{p}\cdot\vec{x}/\hbar}$,

$$Z = \int \mathcal{D}\psi^*(\tau)\mathcal{D}\psi(\tau) \exp\left[-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau \left(\psi^*\dot{\psi} + \psi^* E\psi - \mu\psi^*\psi\right)\right], \quad (1)$$

where $E = \vec{p}^2/2m$. The integration variable $\psi(\tau)$ satisfies the periodic boundary condition $\psi(\hbar\beta) = \psi(0)$. You can Fourier expand it as

$$\psi(\tau) = \sum_{n=-\infty}^{\infty} z_n e^{i2\pi n\tau/\hbar\beta} \quad (2)$$

and do the full path integral using the measure

$$\mathcal{D}\psi^*(\tau)\mathcal{D}\psi(\tau) = \prod_{n=-\infty}^{\infty} dz_n^* dz_n. \quad (3)$$

Show that the result of the partition function is

$$Z = c\frac{e^{-\beta(E-\mu)/2}}{1 - e^{-\beta(E-\mu)}} \quad (4)$$

where c is an overall constant that does not depend on m or β. Then using the fact that $Z = e^{-\beta\Omega}$ with $\Omega = F - \mu N$, work out the expectation value of the number and the energy. The following identity comes in handy:

$$\prod_{n=1}^{\infty}\left(1 + \frac{x^2}{n^2}\right) = \frac{\sinh \pi x}{\pi x} \quad (5)$$

2. Obtain eigenstates of the following Hamiltonian

$$H = \hbar\omega a^\dagger a + Va + V^* a^\dagger \quad (6)$$

for a complex parameter V using the coherent states.