Physics 221B: Solution to HW #10

1) The Electromagnetic Field and its Hamiltonian[1]

a)

This is a standard computation which can be found in most books on quantum field theory, though perhaps in the context of the scalar Klein-Gordon field.

$$H = \frac{1}{8\pi}\int d\vec{x}\ \vec{E}^2 + \vec{B}^2.$$

Using $\vec{E} = -\frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ and plugging in the mode expansion for $\vec{A}$, the $\vec{E}^2$ contribution to the energy is

$$\begin{aligned}\int d\vec{x}\ \vec{E}^2 &= \int d\vec{x}\ \tfrac{1}{c^2}\tfrac{2\pi\hbar c^2}{L^3}\textstyle\sum_{\vec{p},\vec{q},\lambda,\lambda'}(-i)^2\sqrt{\omega_{\vec{p}}\omega_{\vec{q}}}\,(\epsilon^i_\lambda(\vec{p})a_\lambda(\vec{p})e^{i\vec{p}\cdot\vec{x}/\hbar} - \epsilon^i_\lambda(\vec{p})^* a^\dagger_\lambda(\vec{p})e^{-i\vec{p}\cdot\vec{x}/\hbar}) \\ &\quad * (\epsilon^i_{\lambda'}(\vec{q})a_{\lambda'}(\vec{q})e^{i\vec{q}\cdot\vec{x}/\hbar} - \epsilon^i_{\lambda'}(\vec{q})^* a^\dagger_{\lambda'}(\vec{q})e^{-i\vec{q}\cdot\vec{x}/\hbar}).\end{aligned}$$

After multiplying out, rewrite

$$\textstyle\int d\vec{x}\ e^{i(\vec{p}\pm\vec{q})\cdot\vec{x}/\hbar} \rightarrow (2\pi\hbar)^3\delta^3(\vec{p}\pm\vec{q})$$

$$\textstyle\sum_{\vec{q}} \rightarrow \frac{L^3}{(2\pi\hbar)^3}\int d\vec{q}.$$

Then since $\omega_{-\vec{p}} = \omega_{\vec{p}}$, after carrying out the obvious integrals we have

$$\begin{aligned}\int d\vec{x}\ \vec{E}^2 &= -\textstyle\sum_{\vec{p}} 2\pi\hbar\omega_{\vec{p}}\sum_{\lambda,\lambda'}(\epsilon^i_\lambda(\vec{p})a_\lambda(\vec{p})\epsilon^i_{\lambda'}(-\vec{p})a_{\lambda'}(-\vec{p}) - \epsilon^i_\lambda(\vec{p})^* a^\dagger_\lambda(\vec{p})\epsilon^i_{\lambda'}(\vec{p})a_{\lambda'}(\vec{p}) \\ &\quad - \epsilon^i_\lambda(\vec{p})a_\lambda(\vec{p})\epsilon^i_{\lambda'}(\vec{p})^* a^\dagger_{\lambda'}(\vec{p}) + \epsilon^i_\lambda(\vec{p})^* a^\dagger_\lambda(\vec{p})\epsilon^i_{\lambda'}(-\vec{p})^* a^\dagger_{\lambda'}(-\vec{p})).\end{aligned}$$

Now

$$\epsilon^i_\lambda(\vec{p})\epsilon^i_{\lambda'}(-\vec{p}) = -\delta_{\lambda,\lambda'}$$

$$\epsilon^i_\lambda(\vec{p})^*\epsilon^i_{\lambda'}(\vec{p}) = \delta_{\lambda,\lambda'},$$

with analogous results for the other combinations (check simple cases). Then

$$\int d\vec{x}\ \vec{E}^2 = \textstyle\sum_{\vec{p},\lambda} 2\pi\hbar\omega_{\vec{p}}\,(a_\lambda(\vec{p})a_\lambda(-\vec{p}) + a^\dagger_\lambda(\vec{p})a_\lambda(\vec{p}) + a_\lambda(\vec{p})a^\dagger_\lambda(\vec{p}) + a^\dagger_\lambda(\vec{p})a^\dagger_\lambda(-\vec{p})).$$

[1] I thank Ed Boyda once more.

The terms like aa and $a^\dagger a^\dagger$ cancel with similar terms from $\vec{B}^2$ while the other terms add. Including the $1/8\pi$ from the definition of energy,

$$H = \frac{1}{8\pi} \int d\vec{x}\ \vec{E}^2 + \vec{B}^2 = \frac{1}{2} \sum_{\vec{p},\lambda} \hbar\omega_{\vec{p}}\ (a_\lambda^\dagger(\vec{p}) a_\lambda(\vec{p}) + a_\lambda(\vec{p}) a_\lambda^\dagger(\vec{p})).$$

Using $[a, a^\dagger] = 1$ gives the result

$$H = \sum_{\vec{p},\lambda} \hbar\omega_{\vec{p}}\ (a_\lambda^\dagger(\vec{p}) a_\lambda(\vec{p}) + \frac{1}{2}).$$

b)

We consider the coherent state of photons with $\vec{p} = (0, 0, p)$ and helicity $\lambda = +$.

$$|f, t\rangle := e^{-f^* f/2} e^{f e^{-ic|\vec{p}|t/\hbar} a_+^\dagger(\vec{p})} |0\rangle.$$

$$i\hbar \frac{\partial}{\partial t} |f, t\rangle = c\,|\vec{p}|\ f e^{-ic|\vec{p}|t/\hbar} a_+^\dagger(\vec{p}) |f, t\rangle.$$

Since $|f, t\rangle$ is an eigenstate of the annihilation operator, $a_\lambda(\vec{q})|f, t\rangle = \delta_{\lambda +} \delta_{\vec{p}\vec{q}}\ f e^{-ic|\vec{p}|t/\hbar} |f, t\rangle$,

$$H\,|f, t\rangle = \sum_{\vec{q},\lambda} c\,|\vec{q}|\ a_\lambda^\dagger(\vec{q}) a_\lambda(\vec{q}) |f, t\rangle = c\,|\vec{p}|\ a_\lambda^\dagger(\vec{p})\ f e^{-ic|\vec{p}|t/\hbar} |f, t\rangle,$$

ignoring the zero point energy and using the delta functions to perform the sums. Clearly $i\hbar \frac{\partial}{\partial t} |f, t\rangle = H\,|f, t\rangle$.

c)

Again, $|f, t\rangle$ is an eigenstate of the annihilation operator and $\langle f, t|$ is an eigenstate of the creation operator so that

$$\langle f, t|\, a_\lambda(\vec{q}) |f, t\rangle = \delta_{\lambda +} \delta_{\vec{p}\vec{q}}\ f e^{-ic|\vec{p}|t/\hbar},$$

$$\langle f, t|\, a_\lambda^\dagger(\vec{q}) |f, t\rangle = \delta_{\lambda +} \delta_{\vec{p}\vec{q}}\ f^* e^{ic|\vec{p}|t/\hbar}.$$

The definition of $\vec{A}$ gives immediately

$$\langle f, t|\vec{A}|f, t\rangle = \sqrt{\frac{2\pi\hbar c^2}{L^3}} \frac{1}{\sqrt{\omega_{\vec{p}}}} (\vec{\varepsilon}_+(\vec{p}) f e^{-ip\cdot x/\hbar} + \vec{\varepsilon}_+^{\,*}(\vec{p}) f^* e^{ip\cdot x/\hbar}),$$

where $p \cdot x = c\,|\vec{p}|\,t - \vec{p} \cdot \vec{x}$ is the Minkowski scalar product. The coherent state expectation value reproduces a classical plane wave.

3)

a)

Work in units where $\hbar = 1$. It is convenient to rewrite the Hamiltonian as

$$H = -J \sum_{\langle ij \rangle} \vec{s}_1 \cdot \vec{s}_2 = -J \sum_{\langle ij \rangle} s_{zi} s_{zj} + \frac{1}{2}(s_{+,i} s_{-,j} + s_{-,i} s_{+,j})$$

where $s_{\pm} = s_x \pm i s_y$. When all spins are up along the z axis only the first term in H contributes because the other two terms will "try to raise" spins that are already up. Therefore, defining $|0\rangle \equiv |\uparrow\uparrow\uparrow\uparrow \ldots\rangle$

$$H|0\rangle = -J \sum_{\langle i,j \rangle} s_{zi} s_{zj} |0\rangle = -J \sum_{\langle i,j \rangle} \frac{1}{4}|0\rangle = -\frac{NJ}{4}|0\rangle$$

where N the number of pairs.

b)

The system is rotationally invariant, so the Hamiltonian should commute with the rotation operator. We can check this for the particular rotation $\widetilde{U} = \Pi_i U(\theta) = e^{-i\theta \sum_i s_{y_i}}$:

$$\begin{aligned} [s_{y_i} + s_{y_j}, \vec{s}_i \cdot \vec{s}_j] &= [s_{y_i} + s_{y_j}, s_{xi} s_{xj} + s_{y_i} s_{y_j} + s_{zi} s_{zj}] \\ &= -i s_{zi} s_{xj} - i s_{xi} s_{zj} + i s_{xi} s_{zj} + i s_{zi} s_{xj} = 0. \end{aligned}$$

Commuting operators have commuting exponentials, so $H\widetilde{U} = \widetilde{U}H$; the Hamiltonian is invariant under the rotation. This means that the new ground state $|0'\rangle := \widetilde{U}|0\rangle$ satisfies

$$H\widetilde{U}|0\rangle = \widetilde{U}H|0\rangle = E_0 \widetilde{U}|0\rangle,$$

so the rotated state is also a ground state, an equivalent "choice" for the spontaneous symmetry breaking.

We want to check that the two ground states are orthogonal in the limit $N \to \infty$ where N is the number of spins. Consider a given spin which in the ground state is in the state $|\uparrow\rangle = \binom{1}{0}$. The rotation sends this to

$$|\uparrow'\rangle = \begin{pmatrix} \cos\frac{\theta}{2} - \sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} + \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle.$$

Taking the inner product $\langle 0|0'\rangle$ will give a product of factors $\langle\uparrow\,|\uparrow'\rangle$, one for each spin. The factors are

$$\langle\uparrow\,|\uparrow'\rangle = \langle\uparrow|(\cos\frac{\theta}{2}|\uparrow\rangle + \sin\frac{\theta}{2}|\downarrow\rangle) = \cos\frac{\theta}{2}.$$

For N spins,

$$\langle 0|0'\rangle = (\cos\frac{\theta}{2})^N.$$

For any non-zero rotation, the factor $\cos\frac{\theta}{2}$ will be less than one. Thus as $N\to\infty$, $(\cos\frac{\theta}{2})^N \to 0$.

c)

Now we consider the state

$$|\psi\rangle = \sum_n e^{ikna}|\uparrow\uparrow\uparrow\uparrow\downarrow_n\uparrow\uparrow\ldots\rangle.$$

This time when we act H on $|\psi\rangle$ the last two terms in H may contribute. Defining $|\psi_n\rangle \equiv |\uparrow\uparrow\downarrow_n\uparrow\ldots\rangle$ we see how H acts

$$H|\psi_n\rangle = -J\frac{N-4}{4}|\psi_n\rangle - \frac{J}{2}(|\psi_{n-1}\rangle + |\psi_{n+1}\rangle).$$

The first term above is just the groundstate energy form part (a), but after two pairs have changed from $s_{zi}s_{zj} = +1$ to $s_{zi}s_{zj} = -1$. The rest comes from the s_+s_- terms "moving" the spin thats pointing down by one site to the left or to the right.

Now, for $|\psi\rangle = \sum_n e^{inka}|\psi_n\rangle$ we get

$$\begin{aligned}
&H|\psi\rangle = \\
&= -J\sum_n e^{inka}\frac{(N-4)}{4}|\psi_n\rangle - \frac{J}{2}\sum_n e^{i(n+1)ka}|\psi_n\rangle - \frac{J}{2}\sum_n e^{i(n-1)ka}|\psi_n\rangle \\
&= -J\left(\frac{N-4}{4} - \frac{1}{2}e^{ika} - \frac{1}{2}e^{-ika}\right)\sum_n e^{inka}|\psi_n\rangle = -J\left(\frac{N}{4} + 1 - \cos ka\right)|\psi\rangle.
\end{aligned}$$

The excitation energy is obviously

$$\Delta E = J(1-\cos ka).$$

This is a tiny excitation for $N \gg 1$, as we expect.