Physics 221B: Solutions to HW 2

1) a)

Inserting the potential $V(x) = \gamma\delta(x)$ into the Lippmann-Schwinger equation from HW (1) and integrating over the delta-function,

$$\psi(x) = \frac{e^{ikx}}{\sqrt{2\pi\hbar}} + \frac{-im\gamma}{\hbar^2 k} e^{ik|x|}\psi(0). \tag{1}$$

Evaluating this equation at $x = 0$ gives an expression which we can solve for $\psi(0)$,

$$\psi(0) = \frac{1}{\sqrt{2\pi\hbar}} + \frac{-im\gamma}{\hbar^2 k} e^{ik|x|}\psi(0) \Rightarrow \psi(0) = \frac{1}{\sqrt{2\pi\hbar}} \frac{\hbar^2 k}{\hbar^2 k + im\gamma},$$

so that

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}(e^{ikx} - e^{ikr}\frac{im\gamma}{\hbar^2 k + im\gamma}). \tag{2}$$

b)

In the regions $x > 0$ and $x < 0$, the potential vanishes and $\psi(x)$ is just a sum of same-energy plane waves which clearly satisfies the free Schrödinger equation. What's going on at $x = 0$? You might remember the condition for the discontinuity of $\partial_x\psi$ at the location of a delta function from last semester. But if not, lets refresh our memory. We demand our wavefunction ψ will satisfy Schrödinger equation everywhere,

$$-\frac{\hbar^2}{2m}\partial_x^2\psi(x) + V(x)\psi(x) = E\psi(x).$$

If V happens to be a delta function, $V(x) = \gamma\delta(x)$, we can more easily make sense out of this equation at $x = 0$ by integrating both sides over a small vicinity of the origin.

$$\int_{-\epsilon}^{+\epsilon} dx \left[-\frac{\hbar^2}{2m}\partial_x^2 + \gamma\delta(x)\right]\psi(x) = E\int_{-\epsilon}^{+\epsilon}\psi(x)dx.$$

In the limit $\epsilon \to 0$ the right hand side vanishes. We then get

$$\left[-\frac{\hbar^2}{2m}\partial_x\psi(x)\right]_{-0}^{+0} + \gamma\psi(0) = 0$$

or

$$\psi'(x)|_{-0}^{+0} = \frac{2m\gamma}{\hbar^2}\psi(0).$$

Lets check that our solution from a) satisfies this. Reading off of equation (1)

$$\psi'(x) = \begin{cases} \frac{ike^{ikx}}{\sqrt{2\pi\hbar}} + \frac{-im\gamma}{\hbar^2 k} ike^{ikx}\psi(0) & \text{for } x > 0 \\ \frac{ike^{ikx}}{\sqrt{2\pi\hbar}} + \frac{-im\gamma}{\hbar^2 k}(-ik)e^{-ikx}\psi(0) & \text{for } x < 0 \end{cases}$$

and so

$$\psi'(x)|_{-0}^{+0} = \frac{-im\gamma}{\hbar^2 k} ik\psi(0) - \frac{-im\gamma}{\hbar^2 k}(-ik)\psi(0) = \frac{2m\gamma}{\hbar^2}\psi(0).$$

Alternatively, you can check that ψ satisfies the SE by careful differentiation. Notice that

$$\partial_x e^{ik|x|} = ik\text{sign}(x)e^{ik|x|} \implies$$
$$\partial_x^2 e^{ik|x|} = -k^2[\text{sign}(x)]^2 e^{ik|x|} + 2ik\delta(x)e^{ik|x|} = -k^2 e^{ik|x|} + ik\delta(x)$$

where we've used $[\text{sign}(x)]^2 = 1$ and the fact that $f(x)\delta(x) = f(0)\delta(x)$ (which is more precise in an integral, but nonetheless...). So just plugging eq. (2) into the SE will give

$$\begin{aligned}
&-\frac{\hbar^2}{2m}\partial_x^2\psi(x) + \gamma\delta(x)\psi(x) \\
&= -\frac{\hbar^2}{2m}\frac{-k^2}{\sqrt{2\pi\hbar}}(e^{ikx} - e^{ikr}\frac{im\gamma}{\hbar^2 k + im\gamma}) + \frac{\hbar^2}{2m}\frac{1}{\sqrt{2\pi\hbar}}\frac{im\gamma}{\hbar^2 k + im\gamma}2ik\delta(x) \\
&+\frac{1}{\sqrt{2\pi\hbar}}(e^{ikx} - e^{ikr}\frac{im\gamma}{\hbar^2 k + im\gamma})\gamma\delta(x) \\
&= \frac{\hbar^2 k^2}{2m}\psi(x) - \frac{1}{\sqrt{2\pi\hbar}}\frac{\hbar^2\gamma k}{\hbar^2 k + im\gamma}\delta(x) + \frac{1}{\sqrt{2\pi\hbar}}(1 - \frac{im\gamma}{\hbar^2 k + im\gamma})\gamma\delta(x) \\
&= \frac{\hbar^2 k^2}{2m}\psi(x) = E\psi(x).
\end{aligned}$$

c) See Mathematica notebook.

d)

The pole in $f(k', k)$ is at $k = -im\gamma/\hbar^2$, which corresponds to the real energy $E = -m\gamma^2/2\hbar^2$. So we know there exists a stable bound state of that energy. In our derivation of the Lippmann-Schwinger equation, the only place we assume $E > 0$, i.e. a scattering state, is when we add the incoming plane wave $\phi(x)$ to the right hand side to satisfy our boundary conditions. We can do this because a continuum state by definition can have any energy > 0; in particular we can always find a free solution $\phi(x)$ which has the same energy as our scattering state $\psi(x)$. This does not work for bound states which have discrete energies < 0. But if we leave out $\phi(x)$ and fix different boundary conditions, our derivation of the Lippmann-Schwinger equation holds for bound states too. That is, we can read off the bound-state wavefunction from our solution to part (a):

$$\psi_{bound}(x) \sim e^{ikr}$$

will be a bound-state solution when we plug in $k = -im\gamma/\hbar^2$. Boundary conditions for a bound state are that the wavefunction decays at both infinities, which this clearly does for $\gamma < 0$. Normalizing,

$$\psi_{bound} = \sqrt{\frac{-m\gamma}{\hbar^2}} e^{m\gamma r/\hbar^2}.$$

It is easy to check that

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}e^{m\gamma r/\hbar^2} + \gamma\delta(x)e^{m\gamma r/\hbar^2} = \frac{-m\gamma^2}{2\hbar^2}e^{m\gamma r/\hbar^2}.$$

e)

Plugging $V(\vec{x}) = \gamma\delta(\vec{x})$ into the 3-d Lippmann-Schwinger equation gives

$$\psi(\vec{x}) = \frac{1}{(2\pi\hbar)^{3/2}}e^{i\vec{k}\vec{x}} - \frac{2m}{\hbar^2}\frac{e^{ik|\vec{x}|}}{4\pi|\vec{x}|}\gamma\psi(0). \tag{3}$$

To avoid singularity at the origin we require $\psi(0) = 0$, but then the scattering term vanishes, and we are left with the free plane wave

$$\psi(x) = \begin{cases} \frac{1}{(2\pi\hbar)^{3/2}}e^{i\vec{k}\vec{x}}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

the singularity in the potential allowing the discontinuous wavefunction. This is not exactly the planewave we would have gotten if the potential was zero everywhere. In that case we get $\psi(x) = \frac{1}{(2\pi\hbar)^{3/2}}e^{i\vec{k}\vec{x}}$. However, when doing a scattering experiment we certainly won't notice the change, so we can say that a delta function potential does not scatter in 3 dimensions. Note that eq. (3) can still be consistent at the origin only if ψ goes to zero like $\frac{1}{\sqrt{2\pi\hbar}}\frac{4\pi\hbar^2}{2m}|\vec{x}|$. Assuming this is the case, writing eq. (3) for $x \to 0$ indeed gives

$$\psi(0) = \frac{1}{(2\pi\hbar)^{3/2}} - \frac{1}{(2\pi\hbar)^{3/2}} = 0.$$

2) Yukawa potential in the Born approximation

a)

By straight forward integration in the Born approximation (see e.g. lecture notes 2),

$$
\begin{aligned}
f^{(1)}(\vec{k},\vec{k}') &= -\tfrac{1}{4\pi}\tfrac{2m}{\hbar^2}\int d\vec{x}\,V(\vec{x})e^{i\vec{q}\cdot\vec{x}} \\
&= -\tfrac{2m}{\hbar^2}\tfrac{1}{q}\int_0^\infty dr\,r^2 V_0\tfrac{e^{-r/a}}{r}\sin qr \\
&= -\tfrac{2mV_0}{\hbar^2}\tfrac{1}{q}\left(\tfrac{e^{iqr-r/a}}{2i(iq-1/a)} - \tfrac{e^{-iqr-r/a}}{2i(-iq-1/a)}\right)_0^\infty \\
&= -\tfrac{2mV_0}{\hbar^2}\tfrac{1}{q^2+1/a^2},
\end{aligned}
$$

where $q^2 = |\vec{k}-\vec{k}'|^2 = 2k^2(1-\cos\theta)$. Therefore,

$$
\begin{aligned}
\sigma = \int \tfrac{d\sigma}{d\Omega}d\Omega &= \int 2\pi\left(\tfrac{mV_0}{\hbar^2k^2}\right)^2 \tfrac{d\cos\theta}{(1-\cos\theta+1/2k^2a^2)^2} \\
&= 2\pi\left(\tfrac{mV_0}{\hbar^2k^2}\right)^2 \tfrac{1}{1-\cos\theta+1/2k^2a^2}\Big|_{-1}^{1} \\
&= 4\pi a^2\,\tfrac{m^2V_0^2a^2}{\hbar^4}\tfrac{4}{1+4k^2a^2}.
\end{aligned}
$$

b)

As discussed in lecture, we require that the difference between the true wavefunction ψ and the free plane wave ϕ be small where the potential is large. We compute in the Born approximation at the origin (relabling $\vec{x}' \to \vec{x}$):

$$|\psi-\phi| \sim \frac{2m}{\hbar^2}\left|\int d\vec{x}\,\frac{e^{ikr}}{4\pi r}\frac{V_0\,e^{-r/a}}{r}e^{ikz}\right| \ll 1.$$

The integral is much easier to do if you integrate the r variable first (but some people managed it the other way around with the help of Mathematica or a table of integrals).

$$\frac{2m}{\hbar^2}\int 2\pi\,d\cos\theta\,r^2dr\frac{e^{ikr}}{4\pi r}\frac{V_0\,e^{-r/a}}{r}e^{ikr\cos\theta} = \frac{mV_0}{\hbar^2}\int d\cos\theta\frac{-1}{ik+ik\cos\theta-1/a}$$

$$= -\frac{mV_0}{ik\hbar^2}\log\left(\cos\theta+1-1/ika\right)\Big|_{-1}^{1} = -\frac{mV_0}{ik\hbar^2}\log\left(1-2ika\right).$$

The condition for the validity of the Born approximation is

$$\frac{mV_0}{k\hbar^2}\left|\log\left(1-2ika\right)\right| \ll 1.$$

Recalling the the formula for the log of an imaginary number and squaring for later convenience (even though squaring weakens the meaning of $\ll$) we get

$$\frac{m^2 V_0^2}{k^2 \hbar^4} \ll \frac{1}{[\log(1 + 4k^2a^2)]^2 + [\arctan(2ka)]^2}. \tag{4}$$

We can see that its easier satisfying the above for large k, where arctan ceases to increase and k increases faster than its log. Alternatively, for a given momentum we can satisfy (4) by decreasing the strength of the potential V_0 or the potential range a.

c)

The combination appearing on the left hand side of (4) happens to appear in the total cross-section as well. This can be used to get an upper bound on σ.

$$\sigma = 4\pi a^2 \frac{m^2 V_0^2 a^2}{\hbar^4} \frac{4}{1 + 4k^2a^2} \ll 4\pi a^2 \frac{4}{1 + 4k^2a^2} \frac{k^2a^2}{[\log(1 + 4k^2a^2)]^2 + [\arctan(2ka)]^2}$$

If the huge mess that is multiplying $4\pi a^2$, which we will call $f(ka)$, happens to be smaller or equal to 1 then we've proved $\sigma \ll 4\pi a^2$. ($4\pi a^2$ is used as the geometric cross-section, and not just πa^2. As we saw in lecture this can be heuristically justified by thinking about the smearing of the incoming wave-packet compare to the classical particle.)

The Mathematica notebook bellow plots $f(ka)$ for values of $0 < ka < 10$ to show that indeed $f(ka) \leq 1$. Plotting for larger intervals shows the same. Some of you prefered an analytic proof and showed that $f(x) \leq 1$ by showing that $f(0) = 1$ and $f'(x > 0)$ is always negative.

sol2crossec.nb 1

In[1]:= $\texttt{f[x\_] :=} \dfrac{4\,x^2}{1 + 4\,x^2} \dfrac{1}{\texttt{Log}[1 + 4\,x^2]^2 + \texttt{ArcTan}[2\,x]^2}$

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In[3]:= Plot[f[x], {x, 0, 10}]
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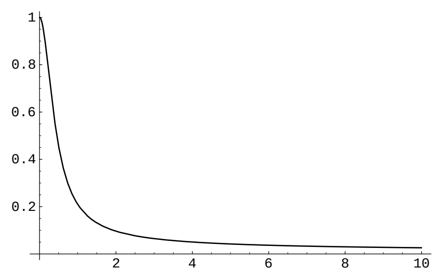


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Out[3]= - Graphics -
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