Physics 221B: Solutions to HW 4

1) Fermions in the Lowest Landau Level and the Fractional Quantum Hall Effect.

a)

Constructing a Slater determinant form $\psi_0 = e^{-(eB/4\hbar c)\bar{z}z}$ and $\psi_1 = ze^{-(eB/4\hbar c)\bar{z}z}$ we get

$$\psi(z_1, z_2) = \begin{vmatrix} \psi_0(z_1) & \psi_0(z_2) \\ \psi_1(z_1) & \psi_1(z_2) \end{vmatrix} = \begin{vmatrix} e^{-(eB/4\hbar c)\bar{z}_1 z_1} & e^{-(eB/4\hbar c)\bar{z}_2 z_2} \\ z_1 e^{-(eB/4\hbar c)\bar{z}_1 z_1} & z_2 e^{-(eB/4\hbar c)\bar{z}_2 z_2} \end{vmatrix}$$
$$= (z_2 - z_1)e^{-(eB/4\hbar c)(\bar{z}_1 z_1 + \bar{z}_2 z_2)}.$$

b)

Lets begin by noting that the Slater determinant we constructed in a) is indeed of the desired form for $N = 2$. Encouraged, we continue by writing an anti-symmetric wave function for N electrons.

$$\psi(z_1, z_2, \ldots, z_N) = \begin{vmatrix} \psi_0(z_1) & \psi_0(z_2) & \cdots & \psi_0(z_N) \\ \psi_1(z_1) & \psi_1(z_2) & \cdots & \psi_1(z_N) \\ \vdots & \vdots & & \vdots \\ \psi_N(z_1) & \psi_N(z_2) & \cdots & \psi_N(z_N) \end{vmatrix}.$$

If a row or a column in a determinant have a common factor it can be pulled out as a multiplying factor. Here we can factor $e^{-(eB/4\hbar c)\bar{z}_i z_i}$ from the i-th column. Doing this for all rows we get

$$\psi(z_1, z_2, \ldots, z_N) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ z_1^2 & z_2^2 & \cdots & z_N^2 \\ \vdots & \vdots & & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} \exp\left(-\frac{eB}{\hbar c} \sum_{i=1}^{N} \bar{z}_i z_i \right).$$

What we have left to show is

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ z_1^2 & z_2^2 & \cdots & z_N^2 \\ \vdots & \vdots & & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} = \prod_{i<j}^{N} (z_i - z_j). \tag{1}$$

Some of you recognized this as the famous Vandermonde determinant [1]. However, for completeness lets prove it.

We will use mathematical induction. Part a) can serve as a check for $N = 2$. Now we will assume equation (1) is true for $N - 1$, and try showing that it works for N. When playing around with determinants, we can always add or subtract a multiple of a row from any other without changing the result. In this case it is useful to do the following. Starting from the last row of the l.h.s of equation (1), we subtract z_N times the row above it. Doing the same for all the rows (but the first, of course) we get

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ z_1^2 & z_2^2 & \cdots & z_N^2 \\ \vdots & \vdots & & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 \\ z_1 - z_N & z_2 - z_N & \cdots & z_{N-1} - z_N & 0 \\ z_1^2 - z_N z_1 & z_2^2 - z_N z_2 & \cdots & z_{N-1}^2 - z_N z_{N-1} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ z_1^{N-1} - z_N z_1^{N-2} & z_2^{N-1} - z_N z_2^{N-2} & \cdots & z_{N-1}^{N-1} - z_N z_{N-1}^{N-2} & 0 \end{vmatrix}$$

Now we can factor out $z_i - z_N$ from the ith row. Since the last column is all zeros but the first entry we get an $(N-1) \times (N-1)$ sub-determinant.

[1] Vandermonde was a French mathematician who lived in Paris during the late 16th century. Since this determinant never appeared in any of his papers it is not clear why it is named after him, even though it is what he is best known for today.

Doing these two steps we get

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ z_1^2 & z_2^2 & \cdots & z_N^2 \\ \vdots & \vdots & & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} =$$

$$= \prod_{i=1}^{N-1}(z_i - z_N) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 - z_N & z_2 - z_N & \cdots & z_{N-1} - z_N \\ z_1^2 - z_N z_1 & z_2^2 - z_N z_2 & \cdots & z_{N-1}^2 - z_N z_{N-1} \\ \vdots & \vdots & & \vdots \\ z_1^{N-2} & z_2^{N-2} & \cdots & z_{N-1}^{N-2} \end{vmatrix}$$

$$= \prod_{i=1}^{N-1}(z_i - z_N) \prod_{i<j}^{N-1}(z_i - z_j) = \prod_{i<j}^{N}(z_i - z_j)$$

where we've used the assumption made earlier.

We have shown

$$\psi(z_1, z_2, \ldots, z_N) = \prod_{i<j}^{N}(z_i - z_j) \exp\left(-\frac{eB}{\hbar c}\sum_{i=1}^{N}\bar{z}_i z_i\right).$$

Another elegant way of proving the Vandermode relation is by expanding the determinant along the first column getting a polynomial in z_1

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_N \\ z_1^2 & z_2^2 & \cdots & z_N \\ \vdots & \vdots & & \vdots \\ z_1^{N-1} & z_2^{N-1} & \cdots & z_N^{N-1} \end{vmatrix} = A z_1^{N-1} + B z_1^{N-2} + \ldots$$

The constants $A, B, \ldots$ are functions of all the other zs. We know that since the determinant vanishes for every $z_1 \to z_i$ for $i \neq 1$, so we know all the roots $N-1$ of the polynomial are $\{z_2, \ldots, z_N\}$. We can rewrite our polynomial as

$$A\prod_{i=2}^{N}(z_1 - z_i)$$

Now we notice that the coefficient A is a just the sub-determinant of the Vandermonde form for $z_2, \ldots, z_n$ and we can complete the proof by induction.

c)

Laughlin's wavefunction [2] is

$$\psi(z_1, z_2, \ldots, z_N) = \prod_{i<j}^{N} (z_i - z_j)^n \exp\left(-\frac{eB}{\hbar c}\sum_{i=1}^{N} \bar{z}_i z_i\right).$$

Before we make sure the wavefunction is anti-symmetric we can restrict n by other considerations. Anti-symmetry will force the wave function to be zero as $z_i \to z_j$ (continuously). This excludes negative n's since they will cause the wave function to blow up as $z_i \to z_j$. Non-integer n's can cause trouble as well because a fractional power is multi-valued (or has branch cuts). We are therefore restricted to positive integers. Finally, we will demand $\psi \to -\psi$ under exchange of two particles. Many of you said immediately that this enforces n to be odd, which is correct, but requires checking that this does not depend on N. For example, if we exchange particles k and l (lets say $k < l$) the term $z_k - z_l$ obviously changes sign. But, this is not the only term in $\prod_{i<j}^{N}(z_i - z_j)^n$ that contains z_k and z_l. Playing around with $\prod_{i<j}^{N}(z_i - z_j)^n$ one can see that any other sign change is compensated by another, so effectively only $z_k - z_l$ changes sign, and an odd n will do the trick.

A simpler way of showing this is writing $\prod_{i<j}^{N}(z_i - z_j)^n = \left(\prod_{i<j}^{N}(z_i - z_j)\right)^n$ and recalling that $\prod_{i<j}^{N}(z_i - z_j)$ is a determinant. Exchanging particles k and l correspond to switching the kth and lth columns of the Vandermonde determinant, resulting in an overall sign change. (that was the motivation of using determinants for antisymmetric wavefunctions). The overall sign change in Laughlin's wavefunction is $(-1)^n$, therefore n is odd.

d)

As explained in the notes on Landau Levels, the highest power of z we can achieve in Laughlin's wavefunction is $n(N-1) \sim nN$. This highest power can be related to the magnetic flux by $nN = e\Phi/\hbar c$. So the number of electrons, N, can be expressed as

$$N = \frac{1}{n}\frac{e\Phi}{\hbar c}.$$

Since $e\Phi/\hbar c$ is the number of available state (see notes), the fractional filling is $1/k$.

[2] Robert Laughlin is giving next week's Oppenheimer lecture.

2)The Helium Atom with Variational Method

As Prof. Jackson said in lecture, referring to this exercise: "Even at my age, I find it is good to solve an integral once in a while."

The Hamiltonian is

$$H_0 = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \frac{Z_0 e^2}{r_1} + \frac{Z_0 e^2}{r_2}$$
$$\Delta H = \frac{e^2}{r_{12}}$$

where $Z_0 = 2$ was given a subscript to distinguish it from the variational parameter Z in the trial wavefunction

$$\psi(r_1, r_2) = N e^{-\frac{Zr_1}{a}} e^{-\frac{Zr_2}{a}}.$$

(I dropped the prime on Z). It is pretty straightforward to find that

$$N = \frac{Z^3}{\pi a^3}$$
$$\langle H_0 \rangle = 2\frac{Z^2 e^2}{2a} - 2\frac{Z Z_0 e^2}{a}$$

We now want to evaluate $\langle \Delta H \rangle$. It may be useful to employ the formula for $\frac{1}{r_{12}}$ in terms of Y_{lm}, but since it will be used in HW 5 (or at least in the solution to it) I'll use the less elegant way this time, just to show it exists. Since we are doing two angular integrations I'll set r_1 in the z direction (and multiply by $4\pi \times 2\pi$) leaving only the θ_2 integral. relabeling $\theta_2 \to \theta$ we get

$$\left\langle \frac{e^2}{r_{12}} \right\rangle =$$
$$= 8\pi^2 N^2 e^2 \int_0^\infty dr_1 \int_0^\infty dr_2 \int_1^{-1} d(\cos\theta) r_1^2 r_2^2 \frac{e^{-\frac{2Z}{a}(r_1+r_2)}}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta}}$$
$$= 8\pi^2 N^2 e^2 \int_0^\infty dr_1 \int_0^\infty dr_2 r_1^2 r_2^2 e^{-\frac{2Z}{a}(r_1+r_2)} \frac{1}{r_1 r_2} \times$$
$$\left[\sqrt{r_1^2 + r_2^2 + 2r_1 r_2} - \sqrt{r_1^2 + r_2^2 - 2r_1 r_2}\right]$$

The term in the square brackets is $(r_1 + r_2) - |r_1 - r_2| = 2r_<$. Noting

$r_</(r_1 r_2) = 1/r_>$ we get

$$
\begin{aligned}
&\left\langle \frac{e^2}{r_{12}} \right\rangle = \\
&= 8\pi^2 N^2 e^2 \int_0^\infty dr_1 \int_0^\infty dr_2 r_1^2 r_2^2 e^{-\frac{2Z}{a}(r_1+r_2)} \frac{2}{r_>} \\
&= 16\pi^2 N^2 e^2 \left(\int_0^\infty dr_1 \int_0^{r_1} r_1 r_2^2 e^{-\frac{2Z}{a}(r_1+r_2)} + \int_0^\infty dr_1 \int_{r_1}^\infty r_1^2 r_2 e^{-\frac{2Z}{a}(r_1+r_2)} \right).
\end{aligned}
$$

We can easily do this integral by hand or with Mathematica to get

$$\left\langle \frac{e^2}{r_{12}} \right\rangle = \frac{5}{8} \frac{Ze^2}{a}.$$

Finally we get

$$E = \langle H \rangle = \frac{e^2}{a} \left(Z^2 - 2Z_0 Z + \frac{5}{8} Z \right) \tag{2}$$

a) $Z = Z_0 = 2$

Plugging $Z = 2$ into (2) we get $E = -2.75\frac{e^2}{a} = -74.8\text{eV}$. This is within 5% of the experimental value of 78.605 eV.

b) Variational Method

Letting Z vary, we can set an upper-bound on the energy by finding the minimal energy.

$$
\begin{aligned}
&0 = \partial_Z E = \frac{e^2}{a} \left(2Z - 2Z_0 + \frac{5}{8} \right) \implies Z = Z_0 - \frac{5}{16} = \frac{27}{16} \\
&E \mid_{Z=\frac{27}{16}} = -77.5\text{eV}.
\end{aligned}
$$

This is much better— within 1.5% of experimental data. The value we get for Z, which is smaller than 2, reflects the screening of the nuclear charge as seen by an electron by the cloud of the other.