Physics 221B: Solution to HW # 8
Quantum Field Theory

1) Bosonic Grand-Partition Function

The solution to this problem is outlined clearly in the beginning of the lecture notes 'Quantum Field Theory II (Bose Systems)' and will not be repeated here. I will just point out the most common mistake. When dealing with the grand canonical ensemble we cannot simply write $U = -\partial_\beta \log Z$ since this gives us $\partial_{beta}\Omega$ which is clearly not what we want. The correct thermodynamic procedure of getting U is explained in the notes. The result comes out to be the intuitive one

$$U = EN.$$

2) A Useful Hamiltonian

We want to "diagonalize" the Hamiltonian. When working with a and $a^\dagger$ operators this basically means trying to write it in the form $H \sim b^\dagger b$ where $[b, b^\dagger] = 1$. Staring at this Hamiltonian for a few minutes shows that it can be written

$$H = \hbar\omega(a^\dagger + \frac{V}{\hbar\omega})(a + \frac{V^*}{\hbar\omega}) - \frac{VV^*}{\hbar\omega},$$

and indeed $[b, b^\dagger] = 1$ for $b = a + \frac{V^*}{\hbar\omega}$ and $b^\dagger$ its complex conjugate. When b and $b^\dagger$ are adjoints and $[b, b^\dagger] = 1$, the states are determined uniquely to be what you expect:

$$|gs\rangle, \quad b^\dagger|gs\rangle, \quad \frac{1}{\sqrt{2!}}b^\dagger b^\dagger|gs\rangle, \quad \ldots, \tag{1}$$

where $b|gs\rangle = 0$. All we need to do is find the local ground state $|gs\rangle$. We know the coherent state $|f\rangle = e^{\frac{-ff^*}{2}}e^{fa^\dagger}$ is an eigenstate of the operator a with eigenvalue f, so if we choose $f = f_0 \equiv -\frac{V^*}{\hbar\omega}$,

$$b\,|f_0\rangle = (a + \frac{V^*}{\hbar\omega})\,|f_0\rangle = (f_0 - \frac{V^*}{\hbar\omega})\,|f_0\rangle = 0.$$

So our ground state is the coherent state with $f = -\frac{V^*}{\hbar\omega}$, the eigenstates are given in (1), and the eigenvalues of the Hamiltonian are

$$E_n = \hbar\omega n - \frac{VV^*}{\hbar\omega}, \qquad n = 0,\ 1,\ 2\ ,\ \ldots.$$