Physics 221B: Solution to HW # 9

1) The Bogliubov Transformation

From the definition

$$b = a\cosh\eta + a^\dagger \sinh\eta, \qquad b^\dagger = a^\dagger\cosh\eta + a\sinh\eta,$$

we can easily see

$$\begin{aligned}
&[b, b^\dagger] = [a, a^\dagger]\cosh^2\eta + [a^\dagger, a]\sinh^2\eta + ([a, a] + [a^\dagger, a^\dagger])\sinh\eta\cosh\eta \\
&= [a, a^\dagger](\cosh^2\eta - \sinh^2\eta) = [a, a^\dagger] = 1.
\end{aligned}$$

This is why the Bogliubov transformation is useful; we've only changed what we mean by creating and annihilating but we've retained the canonical commutation relation.

Our given Hamiltonian is

$$H = \hbar\omega a^\dagger a + \frac{1}{2}V(aa + a^\dagger a^\dagger).$$

Lets look at the simplest Hamiltonian we can construct from the b operators,

$$\begin{aligned}
\xi b^\dagger b &= \xi(a^\dagger a\cosh^2\eta + a^\dagger a^\dagger\cosh\eta\sinh\eta + aa\cosh\eta\sinh\eta + aa^\dagger\sinh^2\eta) \\
&= \xi(a^\dagger a\cosh 2\eta + (a^\dagger a^\dagger + aa)\tfrac{\sinh 2\eta}{2} + \sinh^2\eta),
\end{aligned}$$

where ξ is some constant. Notice that if

$$\xi\sinh 2\eta = V, \qquad \xi\cosh 2\eta = \hbar\omega,$$

we have reproduced our Hamiltonian up to the constant $\sinh^2\eta$. Playing around with hyperbolic trigonometry we find

$$\xi = \sqrt{(\hbar\omega)^2 - V^2}, \qquad \sinh^2\eta = \frac{1}{2}\left(\frac{\hbar\omega}{\sqrt{(\hbar\omega)^2 - V^2}} - 1\right),$$

and the Hamiltonian is

$$H = \sqrt{(\hbar\omega)^2 - V^2}\; b^\dagger b - \frac{1}{2}\hbar\omega + \frac{1}{2}\sqrt{(\hbar\omega)^2 - V^2}.$$

The energy levels are

$$E_n = \sqrt{(\hbar\omega)^2 - V^2}\;(n + \frac{1}{2}) - \frac{1}{2}\hbar\omega.$$

For Hilbert space operators A, B, the Hausdorff formula reads

$$e^B A e^{-B} = A + [B, A] + \frac{1}{2!}[B, [B, A]] + \dots.$$

To check that $b = UaU^{-1}$ for $U = e^{(aa - a^\dagger a^\dagger)\eta/2}$, first note

$$[(aa - a^\dagger a^\dagger)\eta/2, a] = \eta a^\dagger \qquad \text{and} \qquad [(aa - a^\dagger a^\dagger)\eta/2, a^\dagger] = \eta a.$$

Now, use Hausdorff's formula to write

$$UaU^{-1} = a + a^\dagger \eta + a\frac{1}{2!}\eta^2 + \dots$$

where a appears with even powers of η and $a^\dagger$ with odd powers. Factoring a and $a^\dagger$ out and recognizing the power series

$$UaU^{-1} = a \cosh \eta + a^\dagger \sinh \eta = b.$$

2)Superfluid Flow

a)

Look at the time independent wave $\psi = f(r)e^{in\theta}$, where f is real. The number density is $\rho = \psi^*\psi = f(r)^2$. The current density, using the regular definition and the gradient in polar coordinates, is

$$\vec{j} = \frac{\hbar}{2mi}(\psi^*\vec{\nabla}\psi - \vec{\nabla}\psi^*\psi = f(r)^2\frac{n\hbar}{nr}\hat{\theta}.$$

The θ dependence canceled between the two terms. The velocity of the superfluid is

$$\vec{v} = \vec{j}/\rho = \frac{n\hbar}{nr}\hat{\theta}.$$

b)

The equation of motion we have seen in class is

$$i\hbar\dot{\psi} + \frac{\hbar^2\nabla^2}{2m}\psi + \mu\psi - \lambda\psi^*\psi\psi.$$

Since we are looking at a time independent solution the kinetic term vanishes. Using the polar Laplacian and dividing through by $e^{in\theta}$ gives the equation of motion for f

$$\frac{\hbar^2}{2m}\left(f'' + \frac{1}{r}f' - \frac{n^2}{r^2}f\right) + \mu f - \lambda f^3 = 0.$$

Indeed, for $r \to 0$ we keep only the $O(r^{-2})$ term which shows $f(0) = 0$. If we take $r \to \infty$ we can drop the appropriate terms and assume f asymptotes to a constant (meaning $f'' \to 0$) and we get $f(\infty) = \sqrt{\mu/\lambda}$.

c)

The velocity is in the $\hat{\theta}$ direction and therefore circles the origin.

d)

Next we calculate the circulation, $\kappa = \oint \vec{v} \cdot d\vec{l}$. Since $\vec{v}$ is in the $\hat{\theta}$ direction we have $d\vec{l} = (0, rd\theta, 0)$, and we can calculate

$$\kappa = \oint \vec{v} \cdot d\vec{l} = \int_0^{2\pi} \frac{n\hbar}{mr} r d\theta = \frac{2\pi\hbar}{m} n.$$

We should point out that in order for the field $\psi = f(r)e^{in\theta}$ to be single valued, n must be an integer, and therefore the circulation is quantized.