

Gross–Neveu Model

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We solve Problem 11.3 in Peskin–Schroeder.

Gross–Neveu model is given by the Lagrangian density

$$\mathcal{L} = \sum_i^N \bar{\psi}_i i \not{\partial} \psi_i + \frac{1}{2} g_0^2 \left(\sum_i^N \bar{\psi}_i \psi_i \right)^2 \quad (1)$$

for $i = 1, \dots, N$. Note that the sum over i is taken inside the parentheses before taking the square, which is not very clear in the way Peskin–Schroeder writes the Lagrangian density. I’ve put the subscript g_0 to indicate that it is the bare coupling.

This model describes massless spin 1/2 fermions in one spatial dimension with an attractive short-range potential. What the model does is that fermions get bound by the attractive force, and the fermion pair composite condenses to break a discrete Z_2 symmetry. Because of the condensate, fermions acquire a mass. Therefore the dynamics is very similar to what actually happens in four-dimensional QCD (theory of strong interaction) or BCS (Bardeen–Cooper–Schrieffer) theory of superconductivity. Especially in the limit where N is large with $g_0^2 N$ fixed, this result is supposed to be exact. We will see why this is the case in the course of the problem.

(a)

By substituting $\psi_i \rightarrow \gamma^5 \psi_i$, we find

$$\bar{\psi}_i \psi_i = \psi_i^\dagger \gamma^0 \psi_i \rightarrow \psi_i^\dagger \gamma^5 \gamma^0 \gamma^5 \psi_i = -\psi_i^\dagger \gamma^0 \psi_i = -\bar{\psi}_i \psi_i. \quad (2)$$

We used the fact that γ^5 is hermitian and anticommutes with γ^μ . Therefore, the interaction term $(\bar{\psi}_i \psi_i)^2$ remains the same. The kinetic term is also invariant,

$$\bar{\psi}_i i \not{\partial} \psi_i = \psi_i^\dagger \gamma^0 i \gamma^\mu \partial_\mu \psi_i \rightarrow \psi_i^\dagger \gamma^5 \gamma^0 i \gamma^\mu \partial_\mu \gamma^5 \psi_i = \psi_i^\dagger \gamma^0 i \gamma^\mu \partial_\mu \psi_i = \bar{\psi}_i i \not{\partial} \psi_i \quad (3)$$

because γ^5 is anti-commuted twice with gamma matrices. Therefore, $\psi_i \rightarrow \gamma^5 \psi_i$ is a symmetry of the theory. It is actually a Z_2 symmetry: doing it twice is trivial since $(\gamma^5)^2 = 1$.

In addition, this model has $U(N)$ symmetry that rotates the complex basis of fermion fields ψ_i .

(b)

The Dirac field in D -dimension has the mass dimension $(D - 1)/2$. This is because the action is dimensionless (in the unit $\hbar = c = 1$) while the spacetime volume has mass dimension $-D$, and hence the Lagrangian density has mass dimension D . The kinetic term has a single derivative, which has mass dimension one. Therefore the two Dirac field operators have each mass dimension $(D - 1)/2$. In two dimensions $D = 2$, it is $1/2$. Then the four-fermion operator has mass dimension two, which compensates the spacetime integral. Therefore the coupling constant g is dimensionless. In general, an interaction described by a dimensionless constant is renormalizable according to the power counting.

(c)

We show this by a simple Gaussian integral.

$$\begin{aligned} & \int \mathcal{D}\sigma \exp \left[i \int d^2x \left\{ -\sigma \bar{\psi}_i \psi_i - \frac{1}{2g_0^2} \sigma^2 \right\} \right] \\ &= \int \mathcal{D}\sigma \exp \left[i \int d^2x \left\{ -\frac{1}{2g_0^2} (\sigma + g_0^2 \bar{\psi}_i \psi_i)^2 + \frac{1}{2} g_0^2 (\bar{\psi}_i \psi_i)^2 \right\} \right] \\ &= \left(\det \frac{1}{g_0^2} \right)^{-1/2} \exp \left[i \int d^2x \frac{1}{2} g_0^2 (\bar{\psi}_i \psi_i)^2 \right]. \end{aligned} \tag{4}$$

The overall numerical factor $\det \frac{1}{g_0^2}$ (not matter how divergent it may be) is only a numerical factor and we don't care. It drops out from any expectation values.

Note that $\sigma = -g_0^2 \bar{\psi}_i \psi_i$ on average and it can be regarded as the field that describes a fermion bound state.

There is one subtlety when we use the dimensional regularization. The spacetime integral is D -dimensional, and hence the fermion field has the dimension $(D - 1)/2 = \frac{1}{2} - \epsilon$. On the other hand, to keep the coupling constant g_0 dimensionless, the dimension of σ is $D/2 = 1 - \epsilon$. Then their coupling $\sigma \bar{\psi} \psi$ has dimension $2 - 3\epsilon$, which does not yield dimensionless action upto $2 - 2\epsilon$ dimensional spacetime integral. Therefore, we need to modify

the Lagrangian as

$$\int d^{2-2\epsilon}x \left[\bar{\psi}_i i \not{\partial} \psi_i - \mu^\epsilon \sigma \bar{\psi}_i \psi_i - \frac{1}{2g_0^2} \sigma^2 \right] \quad (5)$$

where μ is an arbitrary energy scale (called “renormalization scale”) associated with the dimensional regularization.

(d)

The path integral over the fermion fields yield the determinant

$$\int \mathcal{D}\psi_i \mathcal{D}\bar{\psi}_i \exp \left[i \int d^D x \{ \bar{\psi}_i i \not{\partial} \psi_i - \mu^\epsilon \sigma \bar{\psi}_i \psi_i \} \right] = [\text{Det}(i \not{\partial} - \sigma)]^N. \quad (6)$$

Since we are interested in the effective potential, not the full 1PI effective action, for σ , we can regard $\sigma(x) = \sigma$ to be a spacetime constant. Hence this is nothing but the determinant of N fermions of mass $m = \mu^\epsilon \sigma$. Here, the determinant over the functional space Det including the Dirac indices, and that over the Dirac indices only det is distinguished.

We compute the determinant in the following way. First of all, we use

$$\text{Det}(i \not{\partial} - m) = \exp \text{Tr} \ln(i \not{\partial} - m). \quad (7)$$

In two dimensions,

$$\text{Tr} \ln(i \not{\partial} - m) = \int \frac{d^2x d^2p}{(2\pi)^2} \text{tr} \ln(\not{p} - m) = \int \frac{d^2x d^2p}{(2\pi)^2} \ln(-p^2 + m^2). \quad (8)$$

Here, the trace over the functional space Tr including the Dirac indices, and that over the Dirac indices only tr is distinguished.

The last step can be shown in many ways. One explicit way is using $\gamma^0 = \sigma^2$, $\gamma^1 = i\sigma^1$, and

$$\begin{aligned} \det(\not{p} - m) &= \det \begin{pmatrix} -m & -ip^0 - ip^1 \\ ip^0 - ip^1 & -m \end{pmatrix} \\ &= m^2 - (-ip^0 - ip^1)(ip^0 - ip^1) = m^2 - (p^0)^2 + (p^1)^2 = m^2 - p^2. \end{aligned} \quad (9)$$

Hence,

$$\text{tr} \ln(\not{p} - m) = \ln \det(\not{p} - m) = \ln(-p^2 + m^2). \quad (10)$$

Another way that applies to any dimensions uses the fact that

$$\not{p}^2 = \gamma^\mu \gamma^\nu p_\mu p_\nu = \frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) p_\mu p_\nu = g^{\mu\nu} p_\mu p_\nu = p^2. \quad (11)$$

On the other hand,

$$\text{tr} \not{p} = \text{tr} \gamma^\mu p_\mu = 0. \quad (12)$$

Therefore, in the basis where \not{p} is diagonal, it must be

$$\not{p} = \text{diag}(\sqrt{p^2}, \dots, \sqrt{p^2}, -\sqrt{p^2}, \dots, -\sqrt{p^2}). \quad (13)$$

In two dimensions, Dirac matrices are two-by-two (in D -dimensions, they are $2^{\lfloor D/2 \rfloor}$ -by- $2^{\lfloor D/2 \rfloor}$ where $\lfloor \cdot \rfloor$ is Gauss' symbol for the largest integer less than or equal to the argument). Then we find

$$\text{tr} \ln(\not{p} - m) = 2^{\lfloor D/2 \rfloor - 1} (\ln(\sqrt{p^2} - m) + \ln(-\sqrt{p^2} - m)) = 2^{\lfloor D/2 \rfloor - 1} \ln(-p^2 + m^2). \quad (14)$$

Back to the determinant, we follow Eqs. (11.71,11.72) in Peskin–Schroeder

$$\int \frac{d^D p}{(2\pi)^D} \ln(-p^2 + m^2) = -i \frac{\Gamma(-D/2)}{(4\pi)^{D/2}} (m^2)^{D/2} = -i \frac{\Gamma(-1 + \epsilon)}{(4\pi)^{1-\epsilon}} (m^2)^{1-\epsilon}, \quad (15)$$

where we used $D = 2 - 2\epsilon$. Expanding in ϵ ,

$$\begin{aligned} \text{Tr} \ln(i\not{\partial} - m) &= i \frac{1}{4\pi} \left(\frac{1}{\epsilon} + 1 - \gamma + \ln 4\pi + O(\epsilon) \right) (m^2)^{1-\epsilon} \\ &= i \frac{1}{4\pi} \left[\left(\frac{1}{\epsilon} + 1 - \gamma + \ln 4\pi \right) m^2 - m^2 \ln m^2 + O(\epsilon) \right] \end{aligned} \quad (16)$$

Therefore,

$$\begin{aligned} e^{-i \int d^D x V_{\text{eff}}(\sigma)} &= \int \mathcal{D}\psi_i \mathcal{D}\bar{\psi}_i \exp \left[i \int d^D x \left\{ \bar{\psi}_i i \not{\partial} \psi_i - \mu^\epsilon \sigma \bar{\psi}_i \psi_i - \frac{1}{2g_0^2} \sigma^2 \right\} \right] \\ &= \exp \left[i \int d^D x \left\{ -\frac{1}{2g_0^2} \sigma^2 + \frac{N}{4\pi} \left(\frac{1}{\epsilon} + 1 - \gamma + \ln 4\pi \right) \sigma^2 - \frac{N}{4\pi} \sigma^2 \ln \frac{\sigma^2}{\mu^2} \right\} \right]. \end{aligned} \quad (17)$$

In the $\overline{\text{MS}}$ scheme, we renormalize $\frac{1}{\epsilon} - \gamma + \ln 4\pi$ into the coupling

$$\frac{1}{2g^2} = \frac{1}{2g_0^2} - \frac{N}{4\pi} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi \right), \quad (18)$$

and we obtain the effective potential

$$V_{eff}(\sigma) = \frac{1}{2g^2}\sigma^2 + \frac{N}{4\pi}\sigma^2 \left(\ln \frac{\sigma^2}{\mu^2} - 1 \right). \quad (19)$$

If you employ a different renormalization scheme, the difference is only a change in the definition of $1/g^2 \rightarrow 1/g^2 + c$ by a (finite) constant c . It changes the effective potential to

$$V_{eff}(\sigma) = \frac{1}{2g^2}\sigma^2 + \frac{N}{4\pi}\sigma^2 \left(\ln \frac{\sigma^2}{\mu^2} - 1 + \frac{2\pi}{N}c \right), \quad (20)$$

which can be absorbed into the definition of μ .

Because the physics should be independent of the arbitrary dimensionful scale μ we introduced into the dimensional regularization, we require

$$\mu \frac{\partial}{\partial \mu} V_{eff} = 0, \quad (21)$$

which gives

$$\mu \frac{\partial}{\partial \mu} \frac{1}{g^2} = \frac{N}{\pi}, \quad (22)$$

and hence the coupling is asymptotically free.

We can also calculate the determinant with a sharp momentum cutoff. This makes the correspondence to the Wilsonian view more transparent. We first make a Wick rotation

$$\int \frac{d^2 p}{(2\pi)^2} \ln(-p^2 + m^2) = i \int \frac{d^2 p_E}{(2\pi)^2} \ln(p_E^2 + m^2). \quad (23)$$

Since we are only interested in the $m = \sigma$ dependence of the result, we can subtract an (infinite) constant independent of m ,

$$\begin{aligned} & i \int \frac{d^2 p_E}{(2\pi)^2} [\ln(p_E^2 + m^2) - \ln p_E^2] \\ &= i \int \frac{d^2 p_E}{(2\pi)^2} \int_0^{m^2} d\mu^2 \frac{1}{p_E^2 + \mu^2} \\ &= i \int_0^{m^2} d\mu^2 \frac{1}{4\pi} \ln \frac{\Lambda^2}{\mu^2} \\ &= i \frac{1}{4\pi} m^2 \left(\ln \frac{\Lambda^2}{m^2} + 1 \right). \end{aligned} \quad (24)$$

Therefore, we find the effective potential

$$V_{eff}(\sigma) = \frac{1}{2g_0^2}\sigma^2 + \frac{N}{4\pi}\sigma^2 \left(\ln \frac{\sigma^2}{\Lambda^2} - 1 \right). \quad (25)$$

Here, one can observe the Wilsonian renormalization group evolution of the bare coupling defined with the cutoff Λ . As one change the cutoff Λ by integrating out a slice in the momentum space, the bare coupling constant g_0^2 changes accordingly without changing physics. This gives the cutoff-independence of the effective potential

$$\Lambda \frac{\partial}{\partial \Lambda} V_{eff}(\sigma) = 0, \quad (26)$$

which gives

$$\Lambda \frac{\partial}{\partial \Lambda} \frac{1}{g_0^2} = \frac{N}{\pi}. \quad (27)$$

Again, the coupling is asymptotically free.

(e)

We find the minimum of the effective potential at

$$\frac{dV_{eff}}{d\sigma^2} = \frac{1}{2g^2} + \frac{N}{4\pi} \ln \frac{\sigma^2}{\mu^2} = 0, \quad (28)$$

and hence

$$\sigma^2 = \mu^2 e^{-2\pi/Ng^2}. \quad (29)$$

There are two solutions, $\sigma = \pm \mu e^{-\pi/Ng^2}$, which signals the spontaneous breaking of the \mathbb{Z}_2 symmetry with two ground states. At the minimum, σ is finite and hence the fermions have a finite mass $m = |\sigma| = \mu e^{-\pi/Ng^2}$. The renormalization-scheme-dependent change in $1/g^2$ to $1/g^2 + c$ of course changes the expression by a factor $e^{-\pi c/N}$ but cannot change the fact that the discrete symmetry is broken. σ is the order parameter of this symmetry breaking.

If we use the momentum cutoff, we find instead

$$\sigma^2 = \Lambda^2 e^{-2\pi/Ng_0^2}. \quad (30)$$

This result is an example of the phenomenon called “dimensional transmutation.” The original theory appeared scale-invariant because it did not contain any dimensionful parameters. However, the renormalization forces one to regularize the theory, in our case dimensional regularization, which necessarily makes the theory not scale-invariant and introduces a dimensionful parameter. In the end it develops symmetry breaking at an energy scale exponentially suppressed compared to the cutoff.

It is also worth computing the terms with derivatives in power series in $\partial_\mu\sigma$. The Feynman rule of ψ - ψ - σ vertex is $-i\mu^\epsilon$. Because the fermions have “mass” $m = \sigma$, we compute the two-point function for σ with finite momentum in $D = 2 - 2\epsilon$ dimensions,

$$\begin{aligned}
& -N(-i\mu^\epsilon)^2 \int \frac{d^D k}{(2\pi)^D} \text{tr} \frac{i}{\not{k} - m} \frac{i}{\not{k} + \not{p} - m} \\
&= -N\mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} \int_0^1 dz \frac{\text{tr}(\not{k} + m)(\not{k} + \not{p} + m)}{[k^2 + 2zk \cdot p + zp^2 - m^2]^2} \\
&= -N\mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} \int_0^1 dz \frac{\text{tr}(\not{k} - z\not{p} + m)(\not{k} + (1-z)\not{p} + m)}{[k^2 + z(1-z)p^2 - m^2]^2} \\
&= -N\mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} \int_0^1 dz \frac{2(k^2 - z(1-z)p^2 + m^2)}{[k^2 + z(1-z)p^2 - m^2]^2} \\
&= -2N\mu^{2\epsilon} \int \frac{d^D k}{(2\pi)^D} \int_0^1 dz \left[\frac{1}{k^2 + z(1-z)p^2 - m^2} - 2 \frac{z(1-z)p^2 - m^2}{[k^2 + z(1-z)p^2 - m^2]^2} \right] \\
&= -\frac{2iN}{(4\pi)^{1-\epsilon}} \mu^{2\epsilon} \int_0^1 dz (-\Gamma(\epsilon) + 2\Gamma(1+\epsilon)) [-z(1-z)p^2 + m^2]^{-\epsilon} \\
&= \frac{2iN}{4\pi} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi - 2 - \ln \frac{m^2}{\mu^2} \right) - \frac{2iN}{4\pi} \int_0^1 dz \ln \frac{-z(1-z)p^2 + m^2}{m^2} \quad (31)
\end{aligned}$$

Together with the tree-level two-point function $-\frac{i}{g_0^2}$, we find the total two-point function

$$\begin{aligned}
\Gamma(p^2) &= -\frac{i}{g_0^2} + \frac{2iN}{4\pi} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi - 2 - \ln \frac{m^2}{\mu^2} \right) - \frac{2iN}{4\pi} \int_0^1 dz \ln \frac{-z(1-z)p^2 + m^2}{m^2} \\
&= -\frac{i}{g^2} + \frac{iN}{2\pi} \left(-2 + \frac{2\pi}{Ng^2} \right) - \frac{iN}{2\pi} \int_0^1 dz \ln \frac{-z(1-z)p^2 + m^2}{m^2} \\
&= -\frac{iN}{\pi} - \frac{iN}{2\pi} \int_0^1 dz \ln \frac{-z(1-z)p^2 + m^2}{m^2} \quad (32)
\end{aligned}$$

The two-point function has a zero at $p^2 = 4m^2$ because

$$\int_0^1 dz \ln[-z(1-z)4 + 1] = -2 \quad (33)$$

Namely, there is a pole for the two-point Green’s function $G(p^2) = \Gamma(p^2)^{-1}$ for σ at $p^2 = 4m^2$ and hence σ is a particle of mass $2m$.

We interpret this result that the fermion composite σ has the potential $V_{eff}(\sigma)$ with a finite vacuum expectation value and has a normal kinetic term, and hence propagates as a physical bound state even though it was originally just a convenient tool to rewrite the Lagrangian. It makes sense that the mass of σ is $2m$. The attractive force between fermions is $1/g^2 = (Ng^2)/N$, and in the large N limit with the 't Hooft coupling Ng^2 fixed, the attractive force is $1/N$. Therefore, in the large N limit, the bound state has a mass given by the mass of the constituents, namely $2m$, with negligible binding energy.

Another important point is the piece at the lowest order in p^2

$$\frac{2iN}{4\pi} \int_0^1 dz z(1-z) \frac{p^2}{m^2} = i \frac{N}{12\pi} \frac{p^2}{m^2}. \quad (34)$$

Going back to the coordinate space, it is nothing but the term

$$i \int d^2x \frac{N}{24\pi} \frac{1}{m^2} (\partial_\mu \sigma)^2 \quad (35)$$

in the 1PI effective action $i\Gamma[\sigma]$ with the positive (correct) coefficient for the kinetic term. The fact that it comes with the correct sign is important, because it justifies *a posteriori* a constant σ gives lower energy than a jaggy σ , and hence it was correct to study the effective potential.

(f)

Since the result $Z = \int \mathcal{D}\sigma e^{i\Gamma[\sigma]}$ is still meant to be integrated over the field σ , the above analysis still misses corrections from the σ -loops. However, the above result is exact in the large N limit defined the following way. Consider the limit $N \rightarrow \infty$ while keeping $g_0^2 N$ fixed. The latter is called *'t Hooft coupling*. In this limit, the entire effective action $\Gamma[\sigma]$ is proportional to N :

$$\Gamma[\sigma] = N \left[-\text{Tr} \ln(i\not{\partial} - \sigma) + \frac{1}{2g_0^2 N} \sigma^2 \right]. \quad (36)$$

For large N , the exponent oscillates wildly and hence we can use the steepest descent method, namely picking only the stationary point of the exponent. This way, our analysis looking only for the stationary point of the effective action is justified in the large N limit.