Final Project
The Coleman–Weinberg Potential
Physics 230A, Spring 2007, Hitoshi Murayama

## (a)

The Lagrangian density in the problem is

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 + (D_\mu\phi)^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \frac{\lambda}{6}(\phi^\dagger\phi)^2, \tag{1}$$

where $\phi(x)$ is a complex scalar field in four spacetime dimensions (except for part (h)), and we are interested in the theory when $m^2 = -\mu^2 < 0$. The covariant derivative includes the vector potential

$$D_\mu\phi = (\partial_\mu + ieA_\mu)\phi. \tag{2}$$

I don't know why they chose this funny normalization for the coupling $\lambda$. The Feynman rule for the four-scalar vertex is $-i\frac{2\lambda}{3}$.

First we work out the scalar potential

$$V(\phi) = -\mu^2\phi^\dagger\phi + \frac{\lambda}{6}(\phi^\dagger\phi)^2. \tag{3}$$

The minimum is at

$$\left.\frac{\partial V}{\partial\phi^\dagger}\right|_{\phi=\phi_0} = -\mu^2\phi_0 + \frac{\lambda}{3}\phi_0^\dagger\phi_0^2 = 0, \tag{4}$$

and hence

$$\phi_0^\dagger\phi_0 = \frac{3}{\lambda}\mu^2. \tag{5}$$

Using the gauge invariance of the theory $\phi(x) \to e^{-i\alpha(x)}\phi(x)$, one can always make the vacuum expectation value $\phi_0(x)$ real, positive, and constant,

$$\phi_0 = \sqrt{\frac{3}{\lambda}}\,\mu. \tag{6}$$

Expanding the field around this minimum as $\phi = \phi_0 + \frac{1}{\sqrt{2}}(\sigma + i\pi)$, the potential is

$$
\begin{aligned}
V &= -\frac{\lambda}{3}\phi_0^2\left((\phi_0 + \frac{1}{\sqrt{2}}\sigma)^2 + \frac{1}{2}\pi^2\right) + \frac{\lambda}{6}\left((\phi_0 + \frac{1}{\sqrt{2}}\sigma)^2 + \frac{1}{2}\pi^2\right)^2 \\
&= -\frac{\lambda}{3}\phi_0^2\left(\phi_0^2 + \sqrt{2}\phi_0\sigma + \frac{1}{2}\sigma^2 + \frac{1}{2}\pi^2\right) + \frac{\lambda}{6}\left(\phi_0^2 + \sqrt{2}\phi_0\sigma + \frac{1}{2}\sigma^2 + \frac{1}{2}\pi^2\right)^2 \\
&= -\frac{\lambda}{6}\phi_0^4 + \frac{\lambda}{3}\phi_0^2\sigma^2 + \frac{\lambda}{3\sqrt{2}}\phi_0(\sigma^2 + \pi^2)\sigma + \frac{\lambda}{24}(\sigma^2 + \pi^2)^2. \qquad (7)
\end{aligned}
$$

Next, we work on the kinetic term for the scalar field.

$$
\begin{aligned}
&(D_\mu\phi)^\dagger D^\mu\phi \\
&= \left[\frac{1}{\sqrt{2}}\partial_\mu(\sigma - i\pi) - ieA_\mu\left(\phi_0 + \frac{1}{\sqrt{2}}(\sigma - i\pi)\right)\right] \\
&\quad \left[\frac{1}{\sqrt{2}}\partial_\mu(\sigma + i\pi) + ieA_\mu\left(\phi_0 + \frac{1}{\sqrt{2}}(\sigma + i\pi)\right)\right] \\
&= \frac{1}{2}(\partial_\mu\sigma)^2 + \frac{1}{2}(\partial_\mu\pi)^2 + \frac{e}{\sqrt{2}}\phi_0 A_\mu\partial^\mu\pi + \frac{e}{\sqrt{2}}A_\mu(\sigma\partial^\mu\pi - \pi\partial^\mu\sigma) \\
&\quad + e^2\phi_0^2 A_\mu^2 + \sqrt{2}e^2\phi_0 A_\mu^2\sigma + \frac{1}{2}e^2 A_\mu^2(\sigma^2 + \pi^2) \qquad (8)
\end{aligned}
$$

Therefore, the complete Lagrangian density is

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{4}(F_{\mu\nu})^2 + e^2\phi_0^2 A_\mu^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \frac{\lambda}{3}\phi_0^2\sigma^2 + \frac{1}{2}(\partial_\mu\pi)^2 + \sqrt{2}\, e\phi_0 A_\mu\partial^\mu\pi \\
&\quad + \frac{e}{\sqrt{2}}A_\mu(\sigma\partial^\mu\pi - \pi\partial^\mu\sigma) + \sqrt{2}e^2\phi_0 A_\mu^2\sigma + \frac{1}{2}e^2 A_\mu^2(\sigma^2 + \pi^2) \\
&\quad - \frac{\lambda}{3\sqrt{2}}\phi_0(\sigma^2 + \pi^2)\sigma - \frac{\lambda}{24}(\sigma^2 + \pi^2)^2. \qquad (9)
\end{aligned}
$$

We have removed the "cosmological constant" $-\frac{\lambda}{6}\phi_0^4$. All terms in the first line are quadratic in fields and therefore can be regarded the unperturbed bare Lagrangian, while the second and third lines are cubic or quartic in fields and give interactions among particles which we treat as perturbation. The second term in the first line is the mass term for the gauge field. Namely, the electromagnetism has become short-ranged, with its range givey by the Compton wave length.

## (b)

The problem asks us to use the Landau gauge $\partial_\mu A^\mu = 0$. Then the last term in the first line of Eq. (9) vanishes upon integration by parts. To compute the effective potential for $\phi$, we need to know the mass spectrum as its function. The first line of Eq. (9) tells us that there is a massive vector field of mass $2e^2\phi_0^2$, massive scalar field of mass $\frac{2\lambda}{3}\phi_0^2$, and a massless scalar field. The calculation follows closely that in Chapter 11.4 in the book.

To compute the effective action, we would like to calculate the path integral for a general background $\phi = \phi_{cl}$. For the effective potential, we need to consider a spacetime constant $\phi_{cl}$. Then, thanks to the U(1) gauge invariance, we can take $\phi_{cl}$ to be real without a loss of generality. Fortunately, we have already computed the mass terms for $\phi_0$. Therefore, the vector boson mass for the general $\phi_{cl}$ is simply obtained by replacing $\phi_0$ by $\phi_{cl}$. Namely, we have one massive vector of mass $2e^2\phi_{cl}^2$. For the massive scalar field, we need to reexpand the potential as $\phi_{cl} + \frac{1}{\sqrt{2}}(\sigma + i\pi)$. We find $m_\sigma^2 = \frac{1}{2}V''(\phi_{cl}) = \frac{1}{2}(2m^2 + 2\lambda\phi_{cl}^2) = m^2 + \lambda\phi_{cl}^2$. For $\phi_{cl} = \phi_0$, it recovers the mass we worked out before. $\pi$ has mass $m^2 + \frac{\lambda}{3}\phi_{cl}^2$ which vanishes for $\phi_{cl} = \phi_0$ because of the spontaneously broken U(1) symmetry.

One important point to be careful about is the determinant of the massive vector field in this Lagrangian. Because we have used the Landau gauge $\partial_\mu A^\mu = 0$, we can rewrite the Lagrangian for the vector field as

$$\begin{aligned}
&-\frac{1}{4}(F_{\mu\nu})^2 + e^2\phi_{cl}^2 A_\mu^2 \\
&= -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + e^2\phi_{cl}^2 A_\mu A^\mu \\
&= -\frac{1}{2}\partial_\mu A_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) + e^2\phi_{cl}^2 \delta_\mu^\nu A_\nu A^\mu \\
&= \frac{1}{2}A_\nu(\Box\delta_\mu^\nu - \partial^\nu\partial_\mu + 2e^2\phi_{cl}^2\delta_\mu^\nu)A^\mu \\
&= \frac{1}{2}A_\nu\left[(\Box + 2e^2\phi_{cl}^2)\left(\delta_\mu^\nu - \frac{\partial^\nu\partial_\mu}{\Box}\right) + 2e^2\phi_{cl}^2\frac{\partial^\nu\partial_\mu}{\Box}\right]A^\mu. \qquad (10)
\end{aligned}$$

We used integration by parts and dropped the surface terms, and used the notation $\Box = \partial_\mu\partial^\mu$.

The differetial operators

$$P_T = \delta_\mu^\nu - \frac{\partial^\nu\partial_\mu}{\Box}, \qquad P_L = \frac{\partial^\nu\partial_\mu}{\Box} \qquad (11)$$

are projection operators, because $P_T^2 = P_T$, $P_L^2 = P_L$, $P_T P_L = P_L P_T = 0$. It is easier to see in the momentum space,

$$P_T = \delta_\mu^\nu - \frac{k^\nu k_\mu}{k^2}, \qquad P_L = \frac{k^\nu k_\mu}{k^2}, \tag{12}$$

where one can regard them as four-by-four matrices. $P_T$ has rank three, while $P_L$ rank one. In the Landau gauge $\partial_\mu A^\mu = i k_\mu A^\mu = 0$, $P_L$ vanishes and we are left with only three independent components of $A^\mu$ in the space projected by $P_T$. Therefore, the path integral over the gauge field is equivalent to three independent scalar fields,

$$\int \mathcal{D}A_\mu^T e^{i\int d^4x\left(-\frac{1}{4}(F_{\mu\nu})^2 + e^2\phi_{cl}^2 A_\mu^2\right)} = (\det(\Box + 2e^2\phi_{cl}^2))^{-3/2}. \tag{13}$$

Here, $P_T A_\mu^T = A_\mu^T$ is the remaining three components after taking care of the Landau gauge condition $\partial_\mu A^\mu = 0$.

The path integral over the massive scalar $\sigma$ of course is

$$\int \mathcal{D}\sigma e^{i\int d^4x\left(\frac{1}{2}(\partial_\mu\sigma)^2 - (m^2+\lambda\phi_{cl}^2)\sigma^2\right)} = (\det(\Box + m^2 + \lambda\phi_{cl}^2))^{-1/2}. \tag{14}$$

The path integral over the scalar $\pi$ yields a similar determinant with a mass $m^2 + \frac{1}{3}\lambda\phi_{cl}^2$,

$$\int \mathcal{D}\pi e^{i\int d^4x\left(\frac{1}{2}(\partial_\mu\sigma)^2 - (m^2+\frac{1}{3}\lambda\phi_{cl}^2)\sigma^2\right)} = (\det(\Box + m^2 + \frac{1}{3}\lambda\phi_{cl}^2))^{-1/2}. \tag{15}$$

The effective potential is obtained using the by-now familiar formula (11.71,72)[1]

$$\begin{aligned} \ln\det(\partial^2 + m^2) &= \mathrm{Tr}\log(\partial^2 + m^2) \\ &= \int d^D x \int \frac{d^D k}{(2\pi)^D} \log(-k^2 + m^2) \\ &= -i \int d^D x \frac{\Gamma(-D/2)}{(4\pi)^{D/2}} (m^2)^{D/2}. \end{aligned} \tag{16}$$

[1]Peskin–Schroeder does not use $\Box = \partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2$ because the number of vertices of a box (four) refers to the four-dimensional space time, in much the same way that $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ refers to Laplacian in three-dimensional space. They use $\partial^2 = \partial^\mu\partial_\mu$ for arbitrary dimension $D$.

Here, the expression is analytically continued to $D = 4 - 2\epsilon$ dimensions.

Now we can find the effective potential $e^{-i\int d^D x V(\phi_{cl})}$. We find the one-loop piece

$$\begin{aligned}&\Delta V_{\mathit{eff}}(\phi_{cl})\\&\quad = -\frac{\Gamma(-D/2)}{(4\pi)^{D/2}}\left[\frac{3}{2}(2e^2\phi_{cl}^2)^{D/2} + \frac{1}{2}(m^2+\lambda\phi_{cl}^2)^{D/2} + \frac{1}{2}\left(m^2+\frac{\lambda}{3}\phi_{cl}^2\right)^{D/2}\right].\end{aligned} \tag{17}$$

We later add $V(\phi_{cl})$, the classical potential, to obtain the effective potential.

As usual, we expand around four-dimensions and use $\overline{MS}$ renormalization scheme. We find

$$\begin{aligned}\Delta V_{\mathit{eff}}(\phi_{cl}) &= -\frac{1}{2(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma+\ln 4\pi+\frac{3}{2}\right)\\&\left[\frac{3}{2}(2e^2\phi_{cl}^2)^{2-\epsilon}+\frac{1}{2}(m^2+\lambda\phi_{cl}^2)^{2-\epsilon}+\frac{1}{2}\left(m^2+\frac{\lambda}{3}\phi_{cl}^2\right)^{2-\epsilon}\right]\end{aligned} \tag{18}$$

The counter terms are defined to be $\epsilon \to 0$ limit of the combination $\frac{1}{\epsilon} - \gamma + \ln 4\pi$ in the $\overline{MS}$ scheme,

$$\begin{aligned}V_{ct}(\phi_{cl}) &= -\frac{1}{2(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma+\ln 4\pi\right)M^{-2\epsilon}\\&\left[\frac{3}{2}(2e^2\phi_{cl}^2)^2+\frac{1}{2}(m^2+\lambda\phi_{cl}^2)^2+\frac{1}{2}\left(m^2+\frac{1}{3}\lambda\phi_{cl}^2\right)^2\right].\end{aligned} \tag{19}$$

Here, the renormalization scale $M$ was introduced to ensure the counter terms have the correct dimensions. Note that it has the form of a constant, quadratic term, and quartic term of $\phi_{cl}$ and indeed corresponds to the renormalization of the parameters $m^2$ and $\lambda$ (and cosmological constant) of the original Lagrangian, namely

$$\delta_\lambda = \frac{3}{2(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma+\ln 4\pi\right)\left[3(2e^2)^2+\lambda^2+\frac{\lambda^2}{9}\right]M^{-2\epsilon}, \tag{20}$$

$$\delta_{m^2} = \frac{1}{2(4\pi)^2}\left(\frac{1}{\epsilon}-\gamma+\ln 4\pi\right)\lambda m^2\left[1+\frac{1}{3}\right]M^{-2\epsilon}. \tag{21}$$

Their sum has a regular $\epsilon \to 0$ limit,

$$\Delta V_{eff} + V_{ct} = \frac{1}{64\pi^2}\left[3(2e^2\phi_{cl}^2)^2\left(\ln\frac{2e^2\phi_{cl}^2}{M^2}-\frac{3}{2}\right) + (m^2+\lambda\phi_{cl}^2)^2\left(\ln\frac{m^2+\lambda\phi_{cl}^2}{M^2}-\frac{3}{2}\right) + \left(m^2+\frac{1}{3}\lambda\phi_{cl}^2\right)^2\left(\ln\frac{m^2+\lambda\phi_{cl}^2/3}{M^2}-\frac{3}{2}\right)\right]. \tag{22}$$

It is convenient to define $\bar{M}^2 = M^2 e^{3/2}$ so that

$$\Delta V_{eff} + V_{ct} = \frac{1}{64\pi^2}\left[3(2e^2\phi_{cl}^2)^2\ln\frac{2e^2\phi_{cl}^2}{\bar{M}^2} + (m^2+\lambda\phi_{cl}^2)^2\ln\frac{m^2+\lambda\phi_{cl}^2}{\bar{M}^2} + \left(m^2+\frac{1}{3}\lambda\phi_{cl}^2\right)^2\ln\frac{m^2+\lambda\phi_{cl}^2/3}{\bar{M}^2}\right]. \tag{23}$$

This result could have been obtained by blindingly applying the general formula for the one-loop effective potential

$$\Delta V_{eff}(\phi) = \sum_i \frac{(-1)^F}{64\pi^2} m^4(\phi)\left(\ln\frac{m^2(\phi)}{M^2}-\frac{3}{2}\right) = \sum_i \frac{(-1)^F}{64\pi^2} m^4(\phi)\ln\frac{m^2(\phi)}{\bar{M}^2}\ . \tag{24}$$

Here $i$ refers to each degree of freedom, and $(-1)^F$ is $+1$ for bosons, $-1$ for fermions.

Now we add the tree-level potential to obtain the full one-loop effective potential

$$V_{eff}(\phi_{cl}) = m^2\phi_{cl}^2 + \frac{\lambda}{6}\phi_{cl}^4 + \frac{1}{64\pi^2}\left[3(2e^2\phi_{cl}^2)^2\ln\frac{2e^2\phi_{cl}^2}{\bar{M}^2} + (m^2+\lambda\phi_{cl}^2)^2\ln\frac{m^2+\lambda\phi_{cl}^2}{\bar{M}^2} + \left(m^2+\frac{1}{3}\lambda\phi_{cl}^2\right)^2\ln\frac{m^2+\lambda\phi_{cl}^2/3}{\bar{M}^2}\right]. \tag{25}$$

# (c)

Because the problem tells us to regard $\lambda \approx e^4 \ll 1$, the one-loop piece proportional to $\lambda^2 \approx e^8$ is higher order than the piece $\propto e^4$. Therefore the expression further simplifies to

$$V_{eff}(\phi_{cl}) = \frac{\lambda}{6}\phi_{cl}^4 + \frac{3e^4}{16\pi^2}\phi_{cl}^4\ln\frac{2e^2\phi_{cl}^2}{\bar{M}^2}\ . \tag{26}$$

This potential develops a minimum at

$$\frac{\partial V_{eff}}{\partial \phi_{cl}^2} = \frac{\lambda}{3}\phi_{cl}^2 + \frac{3e^4}{16\pi^2}\left(2\phi_{cl}^2 \ln \frac{2e^2\phi_{cl}^2}{\bar{M}^2} + \phi_{cl}^2\right) = 0, \tag{27}$$

and hence

$$\ln \frac{2e^2\phi_{cl}^2}{\bar{M}^2} = -\frac{1}{2} - \frac{16\pi^2\lambda}{9e^4}. \tag{28}$$

Therefore,

$$\phi_{cl}^2 = \bar{M}^2 \frac{1}{2e^2} E^{-16\pi^2\lambda/9e^4 - 1/2}. \tag{29}$$

Here, I employed Mathematica's notation $E$ for the base of natural logarithm to distinguish it from the electric charge $e$.

This is yet another example of dimensional transmutation. The original theory with $m^2 = 0$ is scale-invariant because there is no dimensionful parameter in the theory. The scale $\bar{M}$ was introduced as the renormalization scale, where $\lambda$ and $e^2$ are "measured." The theory, however, turns out to develop a mass scale $\phi_{cl}$ at the minimum of the potential exponentially suppressed relative to the renormalization scale.

# (d)

We now go back to the effective potential Eq. (25) with $m^2 \neq 0$, but still disregarding the pieces proportional to $\lambda^2 \approx e^8 \ll e^4$.

$$V_{eff}(\phi_{cl}) = m^2\phi_{cl}^2 + \frac{\lambda}{6}\phi_{cl}^4 + \frac{3}{64\pi^2}(2e^2\phi_{cl}^2)^2 \ln \frac{2e^2\phi_{cl}^2}{\bar{M}^2}\ . \tag{30}$$

For large $m^2$ of either sign, the one-loop piece is insignificant in deciding whether the symmetry breaking. However, we have already seen that the symmetry breaking occurs for $m^2 = 0$. The question is what happens when $m^2 > 0$ is small so that the one-loop piece is important.

For this purpose, it is useful to regard the potential as a function of $x = 2e^2\phi_{cl}^2/\bar{M}^2$

$$f(x) = \frac{1}{\bar{M}^4} V(x) = ax + bx^2 + cx^2 \log x. \tag{31}$$

For us, $a = m^2/(2e^2\bar{M}^2)$, $b = \lambda/6(2e^2)^2$, $c = 3/64\pi^2$. We regard $b \approx c$, while $a$ is small to study this region. It is clear that for positive $a$, the potential rises

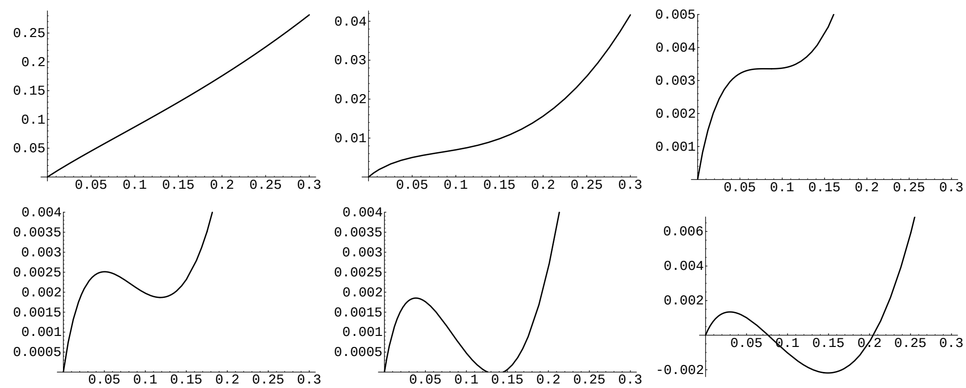


Figure 1: The potential with $b = c = 1$ for $a = 1$, 0.2, 0.164, 0.15, 0.135, 0.12, from the top left to bottom right.

from the origin. However, once $x$ is finite, $x^2 \log x$ gives a negative contribution and can bring the potential down if $a$ is small. Therefore, it overcomes the positive slope at the origin with $a > 0$ and produces a minimum away from the origin.

Just by plotting the potential by Mathematica for various choices of parameters, indeed the potential shows the expected behavior. For example for $b = c = 1$, one can see a symmetry-breaking minimum for $a < 0.135$. Even for larger $a$ up to about 0.164, there is a local minimum where the system may be trapped for a finite lifetime.

Therefore, this potential exhibits a first-order phase transition. Namely that as $m^2$ is lowered from the high temperature, the potential has a well-defined minimum at the origin. But even for $m^2$, it first develops a local minimum away from the origin which comes down as $m^2$ is lowered further. At a critical but positive value of $m^2$, the symmetry-breaking minimum becomes lower than the origin. At this value, the system coexists in both phases, akin to the coexistence of vapor and water at $T = 100°$C. At a lower $m^2$, the symmetry-breaking minimum is the absolute minimum and the system would eventually fall into this minimum. But because of the barrier, the system may be "stuck" at the origin as the system is cooled, and it becomes "supercooled." It is a quantum phenomenon for the system to tunnel to the true minimum.

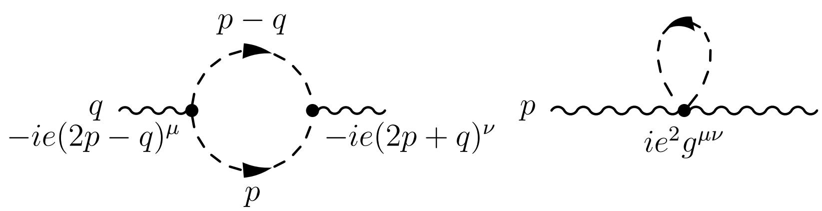


Figure 2: The vacuum polarization diagrams for the photon due to the scalar loop.

# (e)

To work out the beta functions, we can completely ignore $m^2$ for this purpose, as we are only interested in the evolution of dimensionless couplings $e$ and $\lambda$. Therefore, we can work with the complex scalar field $\phi$ as a whole, rather than its real and imaginary parts $\sigma$ and $\pi$.

## (e.1) $\beta_e$

We first study the beta function for the electromagnetic coupling $e$. Because of the Ward identity $Z_1 = Z_2$, we only need to compute the vacuum polarization diagram Fig. 2 and hence $Z_3$.

With the dimensional regularization, the second diagram in Fig. 2 vanishes identically. The first diagram is

$$i\Pi^{\mu\nu}(q) = (-ie)^2 \int \frac{d^Dp}{(2\pi)^D}(2p-q)^\mu(2p-q)^\nu \frac{i}{p^2}\frac{i}{(p-q)^2} \tag{32}$$

Using the by-now-standard methods, we find

$$\begin{aligned}
& i\Pi^{\mu\nu}(q) \\
&= e^2 \int_0^1 dz \int \frac{d^Dp}{(2\pi)^D}\frac{(2p-q)^\mu(2p-q)^\nu}{(p^2-2zp\cdot q+zq^2)^2} \\
&= e^2 \int_0^1 dz \int \frac{d^Dp}{(2\pi)^D}\frac{(2p-(1-2z)q)^\mu(2p-(1-2z)q)^\nu}{(p^2+z(1-z)q^2)^2} \\
&= e^2 \int_0^1 dz \int \frac{d^Dp}{(2\pi)^D}\frac{4p^\mu p^\nu+(1-2z)^2q^\mu q^\nu}{(p^2+z(1-z)q^2)^2} \\
&= e^2 \int_0^1 dz \int \frac{d^Dp}{(2\pi)^D}\frac{\frac{4}{D}g^{\mu\nu}p^2+(1-2z)^2q^\mu q^\nu}{(p^2+z(1-z)q^2)^2}
\end{aligned}$$

$$
\begin{aligned}
&= e^2 \int_0^1 dz \int \frac{d^D p}{(2\pi)^D} \left\{ \frac{\frac{4}{D} g^{\mu\nu}}{p^2 + z(1-z)q^2} + \frac{-\frac{4}{D} g^{\mu\nu} z(1-z)q^2 + (1-2z)^2 q^\mu q^\nu}{(p^2 + z(1-z)q^2)^2} \right\} \\
&= e^2 \int_0^1 dz \frac{i}{(4\pi)^{2-\epsilon}} \left\{ -\Gamma(-1+\epsilon) \frac{4}{D} g^{\mu\nu} [-z(1-z)q^2]^{1-\epsilon} \right. \\
&\quad \left. +\Gamma(\epsilon) \left[ -\frac{4}{D} g^{\mu\nu} z(1-z)q^2 + (1-2z)^2 q^\mu q^\nu \right] [-z(1-z)q^2]^{-\epsilon} \right\} \\
&= e^2 \int_0^1 dz \frac{i}{(4\pi)^{2-\epsilon}} \left\{ -(\Gamma(-1+\epsilon) - \Gamma(\epsilon)) \frac{4}{D} g^{\mu\nu} [-z(1-z)q^2]^{1-\epsilon} \right. \\
&\quad \left. +\Gamma(\epsilon)(1-2z)^2 q^\mu q^\nu [-z(1-z)q^2]^{-\epsilon} \right\} \\
&= e^2 \int_0^1 dz \frac{i}{(4\pi)^{2-\epsilon}} \left\{ -(2-\epsilon)\Gamma(-1+\epsilon) \frac{4}{D} g^{\mu\nu} [-z(1-z)q^2]^{1-\epsilon} \right. \\
&\quad \left. +\Gamma(\epsilon)((1-z)^2 - 2z(1-z) + z^2) q^\mu q^\nu [-z(1-z)q^2]^{-\epsilon} \right\} \\
&= e^2 \frac{i}{(4\pi)^{2-\epsilon}} \left\{ -2\Gamma(-1+\epsilon) g^{\mu\nu} [-q^2]^{1-\epsilon} B(2-\epsilon, 2-\epsilon) \right. \\
&\quad \left. +\Gamma(\epsilon) q^\mu q^\nu [-q^2]^{-\epsilon} (2B(3-\epsilon, 1-\epsilon) - 2B(2-\epsilon, 2-\epsilon) \right\} \\
&= e^2 \frac{i}{(4\pi)^{2-\epsilon}} 2B(2-\epsilon, 2-\epsilon) \Gamma(-1+\epsilon)(q^2 g^{\mu\nu} - q^\mu q^\nu)[-q^2]^{-\epsilon} \\
&= -i \frac{e^2}{(4\pi)^2} \frac{1}{3} \left( \frac{1}{\epsilon} - \gamma + \ln 4\pi + \frac{8}{3} \right) (q^2 g^{\mu\nu} - q^\mu q^\nu)[-q^2]^{-\epsilon} \\
&= i\Pi(q^2)(q^2 g^{\mu\nu} - q^\mu q^\nu). \qquad (33)
\end{aligned}
$$

This is added to the photon propagator in the usual way,

$$
\begin{aligned}
&\quad (-ie) \frac{-ig^{\mu\nu}}{q^2} (-ie) + (-ie) \frac{-ig^{\mu\rho}}{q^2} i\Pi(q^2)(q^2 g_{\rho\sigma} - q_\rho q_\sigma) \frac{-ig^{\sigma\nu}}{q^2} (-ie) + \cdots \\
&= \frac{ig^{\mu\nu} e^2}{q^2(1 - \Pi(q^2))} \\
&= \frac{ig^{\mu\nu} e^2}{q^2 \left[ 1 + \frac{e^2}{(4\pi)^2} \frac{1}{3} \left( \frac{1}{\epsilon} - \gamma + \ln 4\pi + \frac{8}{3} \right) [-q^2]^{-\epsilon} \right]} \\
&= \left. \frac{ig^{\mu\nu} e^2(M)}{q^2} \right|_{q^2 = -M^2} + \text{finite}. \qquad (34)
\end{aligned}
$$

Therefore, the effective electric charge depends on the momentum scale,

$$
e^2(M) = \frac{e^2}{1 + \frac{e^2}{(4\pi)^2} \frac{1}{3} \left( \frac{1}{\epsilon} - \gamma + \ln 4\pi + \frac{8}{3} \right) M^{-2\epsilon}} \qquad (35)
$$

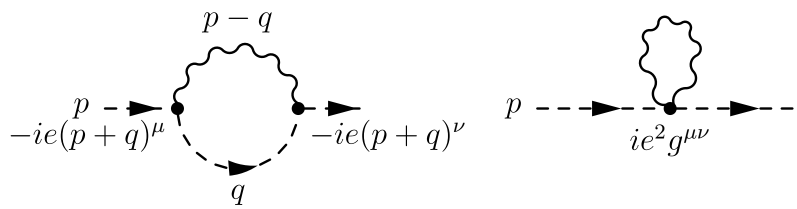


Figure 3: The 1PI two-point function of scalars due to the gauge interaction.

Fixing the bare coupling $e^2$, we find

$$\beta_e = M\frac{de}{dM} = \frac{e^3}{48\pi^2} , \tag{36}$$

as stated in the problem.

## (e.2) $\gamma$

In order to compute the beta function of the coupling $\lambda$, we need to know the wave function renormalzation of the scalar field. As we discussed in the class, there is no wave function renormalization for the scalar field due to the $\phi^4$ interaction at the one-loop level. Therefore, we only need to consider the gauge interation.

The second diagram in Fig. 3 does not contribute to the wave function renormalization and hence is not important to us. With the dimensional regularization, actually the second diagram identically vanishes

$$i\Sigma_2 = \int \frac{d^D q}{(2\pi)^D} ie^2 g_{\mu\nu} \frac{-i}{q^2}\left(g^{\mu\nu} - (1-\xi)\frac{q^\mu q^\nu}{q^2}\right) = e^2 \int \frac{d^D q}{(2\pi)^D}\frac{D-1+\xi}{q^2} = 0. \tag{37}$$

For the purpose of computing the wave function renormalization, we do not need to keep the mass of the scalar finite; it simplifies the calculation. From the first diagram, we find the contribution to the two-point function

$$i\Sigma_1 = (-ie)^2 \int \frac{d^D q}{(2\pi)^D}(p+q)_\mu \frac{i}{q^2}(p+q)_\nu \frac{-i}{(p-q)^2}\left[g^{\mu\nu} - (1-\xi)\frac{(p-q)^\mu (p-q)^\nu}{(p-q)^2}\right]. \tag{38}$$

We first compute the piece without $(1-\xi)$. We find

$$-e^2 \int \frac{d^D q}{(2\pi)^D}\frac{(p+q)^2}{q^2(p-q)^2}$$

$$
\begin{aligned}
&= -e^2 \int_0^1 dz \int \frac{d^D q}{(2\pi)^D} \frac{(p+q)^2}{(q^2 - 2zp\cdot q + zp^2)^2} \\
&= -e^2 \int_0^1 dz \int \frac{d^D q}{(2\pi)^D} \frac{(q+(1+z)p)^2}{(q^2 + z(1-z)p^2)^2} \\
&= -e^2 \int_0^1 dz \int \frac{d^D q}{(2\pi)^D} \frac{q^2 + (1+z)^2 p^2}{(q^2 + z(1-z)p^2)^2} \\
&= -e^2 \int_0^1 dz \int \frac{d^D q}{(2\pi)^D} \left( \frac{1}{q^2 + z(1-z)p^2} + \frac{(1+z+2z^2)p^2}{(q^2+z(1-z)p^2)^2} \right) \\
&= -e^2 \int_0^1 dz \frac{i}{(4\pi)^{2-\epsilon}} \\
&\quad \left( -\frac{\Gamma(-1+\epsilon)}{\Gamma(1)} [-z(1-z)p^2]^{1-\epsilon} + \frac{\Gamma(\epsilon)}{\Gamma(2)} (1+z+2z^2) p^2 [-z(1-z)p^2]^{-\epsilon} \right) \\
&= -e^2 \int_0^1 dz \frac{i}{(4\pi)^{2-\epsilon}} [-p^2]^{1-\epsilon} \\
&\quad \left( -\frac{\Gamma(-1+\epsilon)}{\Gamma(1)} [z(1-z)]^{1-\epsilon} - \frac{\Gamma(\epsilon)}{\Gamma(2)} ((1-z)^2 + 3z + z^2)[z(1-z)]^{-\epsilon} \right) \\
&= -e^2 \frac{i}{(4\pi)^{2-\epsilon}} [-p^2]^{1-\epsilon} \left( -\frac{\Gamma(-1+\epsilon)}{\Gamma(1)} B(2-\epsilon, 2-\epsilon) \right. \\
&\quad \left. -\frac{\Gamma(\epsilon)}{\Gamma(2)} [B(1-\epsilon, 3-\epsilon) + 3B(2-\epsilon, 1-\epsilon) + B(3-\epsilon, 1-\epsilon)] \right) \\
&= -e^2 \frac{i}{(4\pi)^{2-\epsilon}} [-p^2]^{1-\epsilon} \left( -\frac{2}{\epsilon} + 4 + 2\gamma + O(\epsilon) \right) \\
&= -i \frac{2e^2}{(4\pi)^2} p^2 [-p^2]^{-\epsilon} \left( \frac{1}{\epsilon} + 2 - \gamma + O(\epsilon) \right). \qquad (39)
\end{aligned}
$$

The piece proportional to $(1-\xi)$ is a little more complicated. Instead of working it out completely, we just extract the pole piece. We use the Feynman integral

$$
\frac{1}{a^2 b} = -\frac{\partial}{\partial a} \frac{1}{ab} = -\frac{\partial}{\partial a} \int_0^1 dz \frac{1}{[za + (1-z)b]^2} = \int_0^1 dz \frac{2z}{[za + (1-z)b]^3} \;. \qquad (40)
$$

We obtain

$$
e^2 (1-\xi) \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2} \frac{1}{(p-q)^2} \frac{(p^2 - q^2)^2}{(p-q)^2}
$$

$$
\begin{aligned}
&= e^2(1-\xi)\int_0^1 dz \int \frac{d^Dq}{(2\pi)^D}\frac{2z(p^2-q^2)^2}{[q^2-2zp\cdot q+zp^2]^3} \\
&= e^2(1-\xi)\int_0^1 dz \int \frac{d^Dq}{(2\pi)^D}\frac{2z((q+zp)^2-p^2)^2}{[q^2+z(1-z)p^2]^3} \\
&= e^2(1-\xi)\int_0^1 dz \int \frac{d^Dq}{(2\pi)^D}\frac{2z(q^2+2zp\cdot q-(1-z^2)p^2)^2}{[q^2+z(1-z)p^2]^3} \\
&= e^2(1-\xi)\int_0^1 dz \int \frac{d^Dq}{(2\pi)^D}\frac{2z((q^2)^2+4z^2(p\cdot q)^2-2(1-z^2)p^2q^2+\text{finite})}{[q^2+z(1-z)p^2]^3} \\
&= e^2(1-\xi)\int_0^1 dz \int \frac{d^Dq}{(2\pi)^D}\frac{2z((q^2)^2+(\frac{4}{D}z^2-2(1-z^2))p^2q^2+\text{finite})}{[q^2+z(1-z)p^2]^3} \\
&= e^2(1-\xi)\int_0^1 dz\frac{i}{(4\pi)^2}\ 2z\left\{\frac{3}{\epsilon}[-z(1-z)p^2]^{1-\epsilon}\right. \\
&\quad \left.+\left(\frac{4}{D}z^2-2(1-z^2)\right)p^2\frac{1}{\epsilon}[-z(1-z)p^2]^{-\epsilon}+\text{finite}\right\} \\
&= e^2(1-\xi)\frac{2i}{(4\pi)^2}\int_0^1 dz\left\{-\frac{3}{\epsilon}(z^2-z^3)p^2[-p^2]^{-\epsilon}\right. \\
&\quad \left.+\left(3z^3-2z\right)p^2\frac{1}{\epsilon}[-p^2]^{-\epsilon}+\text{finite}\right\} \\
&= e^2(1-\xi)\frac{2i}{(4\pi)^2}\left\{-\frac{3}{\epsilon}\frac{1}{12}p^2[-p^2]^{-\epsilon}-\frac{1}{4}p^2\frac{1}{\epsilon}[-p^2]^{-\epsilon}+\text{finite}\right\} \\
&= -e^2(1-\xi)\frac{i}{(4\pi)^2}\frac{1}{\epsilon}p^2[-p^2]^{-\epsilon}. \qquad (41)
\end{aligned}
$$

Adding both contributions, we find

$$
i\Sigma = i\frac{e^2}{(4\pi)^2}\frac{1}{\epsilon}p^2[-p^2]^{-\epsilon}(-2-(1-\xi)) = -i\frac{e^2}{(4\pi)^2}\frac{1}{\epsilon}p^2[-p^2]^{-\epsilon}(3-\xi). \qquad (42)
$$

Together with the tree-level piece $ip^2$, the 1PI two-point function up to the one-loop order is

$$
i\Gamma^{(2)}(p^2) = -ip^2+i\Sigma_1 = -ip^2\left(1+\frac{e^2}{(4\pi)^2}\frac{1}{\epsilon}(3-\xi)[-p^2]^{-\epsilon}\right)+\text{regular}. \qquad (43)
$$

We impose the renormalization condition

$$
iG^{(2)}(p^2=-M^2) = \frac{1}{i\Gamma^{(2)}(p^2=-M^2)} = \left.\frac{iZ_2(M)}{p^2}\right|_{p^2=-M^2}, \qquad (44)
$$

and we find the wavefunction renormalization factor

$$Z(M) = 1 + \frac{e^2}{(4\pi)^2}\frac{1}{\epsilon}(3-\xi)M^{-2\epsilon}. \tag{45}$$

Therefore,

$$\gamma = \frac{1}{2}M\frac{d}{dM}\ln Z = -\frac{e^2}{(4\pi)^2}(3-\xi). \tag{46}$$

For the Landau gauge, $\xi = 0$ and hence $\gamma = -\frac{3e^2}{(4\pi)^2}$.

## (e.3) $\beta_\lambda$

Of course we can compute the three-point and four-point functions to work out the beta functions. But we can do it in a much easier way using Callan–Symanzik equation (Peskin–Schroeder (13.25)) on the effective potential Eq. (25) with $m^2 = 0$,

$$\left[M\frac{\partial}{\partial M} + \beta_\lambda\frac{\partial}{\partial\lambda} + \beta_e\frac{\partial}{\partial e} - \gamma\phi_{cl}\frac{\partial}{\partial\phi_{cl}}\right]V(\phi_{cl}) = 0. \tag{47}$$

Each term of the equation is

$$M\frac{\partial}{\partial M}V(\phi_{cl}) = \frac{-2}{64\pi^2}\left[3(2e^2\phi_{cl}^2)^2 + (\lambda\phi_{cl}^2)^2 + \left(\frac{1}{3}\lambda\phi_{cl}^2\right)^2\right], \tag{48}$$

$$\beta_\lambda\frac{\partial}{\partial\lambda}V(\phi_{cl}) = \beta_\lambda\frac{1}{6}\phi_{cl}^4 + \text{higher orders}, \tag{49}$$

$$\beta_e\frac{\partial}{\partial e}V(\phi_{cl}) = \text{higher orders}, \tag{50}$$

$$\gamma\phi_{cl}\frac{\partial}{\partial\phi_{cl}}V_{\phi_{cl}} = \gamma\frac{2\lambda}{3}\phi_{cl}^4 + \text{higher orders}. \tag{51}$$

We find

$$\beta_\lambda = \frac{12}{64\pi^2}\left[3(2e^2)^2 + (\lambda)^2 + \left(\frac{1}{3}\lambda\right)^2\right] + 6\frac{2\lambda}{3}\gamma = \frac{1}{24\pi^2}\left[54e^4 + 5\lambda^2 - 18\lambda e^2\right]. \tag{52}$$

## (e.4) Brute Force

We can of course compute the beta function by brute force Feynman diagram calculations. We use the bare perturbation theory, and the coupling constants $e$ and $\lambda$ are the bare couplings with $2\epsilon$ dimensions.

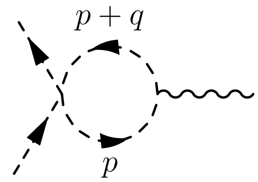


Figure 4: The 1PI three-point function of scalars and a photon due to the $\phi^4$ interaction. It identically vanishes.

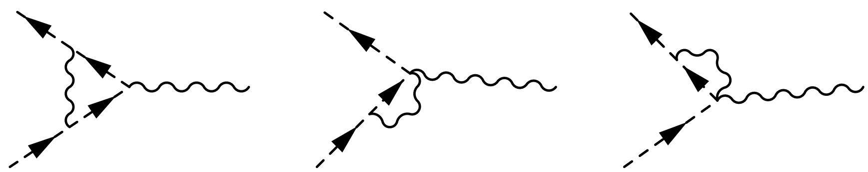

Figure 5: The 1PI three-point function of scalars and a photon at $O(e^3)$.

### (e.4.1) $O(e\lambda)$ Vertex Correction

The vertex correction to $e$ at $O(e\lambda)$ is given in Fig. 4. However, we know it should vanish because it would give a contribution to $Z_1$ at $O(\lambda)$, but there is no contribution to $Z_2$ at $O(\lambda)$ to satisfy the Ward identity $Z_1 = Z_2$. We can verify that it vanishes identically. In other words,

$$\begin{aligned} & -i\frac{2\lambda}{3}(-ie)\int \frac{d^D p}{(2\pi)^D}(2p+q)^\mu \frac{i}{p^2}\frac{i}{(p+q)^2} \\ = & \frac{2e\lambda}{3}\int \frac{d^D p}{(2\pi)^D}\int_0^1 dz \frac{(2p+q)^\mu}{(p^2+2zp\cdot q+zq^2)^2} \\ = & \frac{2e\lambda}{3}\int \frac{d^D p}{(2\pi)^D}\int_0^1 dz \frac{(2p+(1-2z)q)^\mu}{(p^2+z(1-z)q^2)^2} \\ = & \frac{2e\lambda}{3}\int \frac{d^D p}{(2\pi)^D}\int_0^1 dz \frac{(1-2z)q^\mu}{(p^2+z(1-z)q^2)^2} \end{aligned} \tag{53}$$

The point is that the numerator $(1-2z)$ is odd under the change of variable $z \to 1-z$, but the denominator is even. Therefore the result vanishes upon $z$-integration.

### (e.4.2) $O(e^3)$ Vertex Correction

The vertex correction to the electromagnetic coupling at $O(e^3)$ comes from the diagrams in Fig. 5. Again the Ward identity requires $Z_1 = Z_2$, and we have computed the latter already. We compute the divergent parts only. The first diagram without the $(1-\xi)$ piece is

$$
\begin{aligned}
&(-ie)^3 \int \frac{d^D k}{(2\pi)^D} (p+p'+2k)^\mu (2p+k)^\nu (2p'+k)^\rho \frac{i}{(p+k)^2} \frac{i}{(p'+k)^2} \frac{-ig_{\nu\rho}}{k^2} \\
&\quad = -e^3 \int \frac{d^D k}{(2\pi)^D} (p'+p+2k)^\mu \frac{(2p+k)\cdot(2p'+k)}{(p+k)^2 (p'+k)^2 k^2} \\
&\quad = -e^3 2 \int_0^1 d^3 z \delta(1-\sum_i z_i) \frac{d^D k}{(2\pi)^D} \frac{(p+p'+2k)^\mu (2p+k)\cdot(2p'+k)}{[k^2+2z_1 p\cdot k + 2z_2 p'\cdot k + z_1 p^2 + z_2 p'^2]^3} \\
&\quad = -e^3 2 \int_0^1 d^3 z \delta(1-\sum_i z_i) \frac{d^D k}{(2\pi)^D} ((1-2z_1)p + (1-2z_2)p' + 2k)^\mu \\
&\quad\quad \frac{((2-z_1)p - z_2 p' + k)\cdot((2-z_2)p' - z_1 p + k)}{[k^2 + z_1(1-z_1)p^2 + z_2(1-z_2)p'^2 - 2z_1 z_2 p\cdot p']^3} \\
&\quad = -e^3 2 \int_0^1 d^3 z \delta(1-\sum_i z_i) \frac{d^D k}{(2\pi)^D} \\
&\quad\quad \frac{((1-2z_1)p + (1-2z_2)p')^\mu k^2 + k^\mu((2-2z_1)p + (2-2z_2)p')\cdot k + \text{finite}}{[k^2 + z_1(1-z_1)p^2 + z_2(1-z_2)p'^2 - 2z_1 z_2 p\cdot p']^3} \\
&\quad = -e^3 2 \int_0^1 d^3 z \delta(1-\sum_i z_i)[(2-3z_1)p + (2-3z_2)p']^\mu \frac{i}{(4\pi)^2}\Gamma(\epsilon) + \text{finite} \\
&\quad = -e^3 (p+p')^\mu \frac{i}{(4\pi)^2}\Gamma(\epsilon) + \text{finite}. \qquad (54)
\end{aligned}
$$

The gauge-dependent piece is

$$
\begin{aligned}
&(-ie)^3 \int \frac{d^D k}{(2\pi)^D} (p+p'+2k)^\mu (2p+k)^\nu (2p'+k)^\rho \frac{i}{(p+k)^2} \frac{i}{(p'+k)^2} \frac{ik_\nu k_\rho}{k^2} \frac{1-\xi}{k^2} \\
&\quad = e^3 (1-\xi) \int \frac{d^D k}{(2\pi)^D} (p'+p+2k)^\mu \frac{(2p+k)\cdot k\ (2p'+k)\cdot k}{(p+k)^2 (p'+k)^2 (k^2)^2} \\
&\quad = e^3 (1-\xi) 6 \int_0^1 d^3 z \delta(1-\sum_i z_i) \frac{d^D k}{(2\pi)^D} \frac{z_3 (p+p'+2k)^\mu (2p+k)\cdot k\ (2p'+k)\cdot k}{[k^2+2z_1 p\cdot k + 2z_2 p'\cdot k + z_1 p^2 + z_2 p'^2]^4} \\
&\quad = e^3 (1-\xi) 6 \int_0^1 d^3 z \delta(1-\sum_i z_i) \frac{d^D k}{(2\pi)^D} z_3 ((1-2z_1)p + (1-2z_2)p' + 2k)^\mu \\
&\quad\quad \frac{((2-z_1)p - z_2 p' + k)\cdot(k - z_1 p - z_2 p')\ ((2-z_2)p' - z_1 p + k)\cdot(k - z_1 p - z_2 p')}{[k^2 + z_1(1-z_1)p^2 + z_2(1-z_2)p'^2 - 2z_1 z_2 p\cdot p']^4}
\end{aligned}
$$

$$
\begin{aligned}
&= e^3(1-\xi)6\int_0^1 d^3z\delta(1-\sum_i z_i)\frac{d^Dk}{(2\pi)^D}z_3((1-2z_1)p+(1-2z_2)p'+2k)^\mu \\
&\quad \frac{(k^2+(2-2z_1)p\cdot k-2z_2k\cdot p')(k^2-2z_1p+(2-2z_2)p')+\text{finite}}{[k^2+z_1(1-z_1)p^2+z_2(1-z_2)p'^2-2z_1z_2p\cdot p']^4} \\
&= e^3(1-\xi)6\int_0^1 d^3z\delta(1-\sum_i z_i)\frac{d^Dk}{(2\pi)^D}z_3 \\
&\quad \frac{((1-2z_1)p+(1-2z_2)p')^\mu(k^2)^2+2k^\mu k^2((2-4z_1)p+(2-4z_2)p')\cdot k+\text{finite}}{[k^2+z_1(1-z_1)p^2+z_2(1-z_2)p'^2-2z_1z_2p\cdot p']^4} \\
&= e^3(1-\xi)6\int_0^1 d^3z\delta(1-\sum_i z_i)\frac{d^Dk}{(2\pi)^D}z_3 \\
&\quad \frac{((1-2z_1)p+(1-2z_2)p')^\mu+((1-2z_1)p+(1-2z_2)p')^\mu}{[k^2]^2}+\text{finite} \\
&= e^3(1-\xi)6\int_0^1 d^3z\delta(1-\sum_i z_i)z_3\frac{i}{(4\pi)^2}\Gamma(\epsilon)2[(1-2z_1)p+(1-2z_2)p']^\mu+\text{finite} \\
&= e^3(1-\xi)\frac{i}{(4\pi)^2}\Gamma(\epsilon)2(p+p')^\mu 6\int_0^1 dz_1\int_0^{1-z_1}dz_3(1-2z_1)z_3 \\
&= e^3(1-\xi)\frac{i}{(4\pi)^2}\Gamma(\epsilon)(p+p')^\mu 6\int_0^1 dz_1(1-2z_1)(1-z_1)^2 \\
&= e^3(1-\xi)\frac{i}{(4\pi)^2}\Gamma(\epsilon)(p+p')^\mu. \qquad (55)
\end{aligned}
$$

The second diagram is

$$
\begin{aligned}
&2ie^2g^{\mu\nu}(-ie)\int\frac{d^Dk}{(2\pi)^D}(2p+k)^\rho\frac{-ig_{\nu\rho}}{k^2}\frac{i}{(p+k)^2} \\
&\quad = 2e^3\int\frac{d^Dk}{(2\pi)^D}\frac{(2p+k)^\mu}{k^2(p+k)^2} \\
&\quad = 2e^3\int\frac{d^Dk}{(2\pi)^D}\int_0^1 dz\frac{[(2-z)p+k]^\mu}{[k^2+z(1-z)p^2]^2} \\
&\quad = 2e^3\int_0^1 dz(2-z)p^\mu\frac{i}{(4\pi)^2}\Gamma(\epsilon)[-z(1-z)p^2]^{-\epsilon} \\
&\quad = e^3 3p^\mu\frac{i}{(4\pi)^2}\Gamma(\epsilon)+\text{finite}. \qquad (56)
\end{aligned}
$$

The gauge-dependent piece is

$$
\begin{aligned}
&2ie^2 g^{\mu\nu}(-ie)\int \frac{d^Dk}{(2\pi)^D}(2p+k)^\rho \frac{ik_\nu k_\rho}{k^2}\frac{1-\xi}{k^2}\frac{i}{(p+k)^2} \\
&= -2e^3(1-\xi)\int \frac{d^Dk}{(2\pi)^D}\frac{k^\mu k\cdot(2p+k)}{[k^2]^2(p+k)^2} \\
&= e^3(1-\xi)\int \frac{d^Dk}{(2\pi)^D}2\int_0^1 dz \frac{(1-z)k^\mu k\cdot(2p+k)}{[k^2+2zp\cdot k+zp^2]^3} \\
&= e^3(1-\xi)\int \frac{d^Dk}{(2\pi)^D}2\int_0^1 dz \frac{(1-z)(k-zp)^\mu (k-zp)\cdot((2-z)p+k)}{[k^2+z(1-z)p^2]^3} \\
&= e^3(1-\xi)\int \frac{d^Dk}{(2\pi)^D}2\int_0^1 dz \frac{(1-z)(k-zp)^\mu (k^2+2(1-z)k\cdot p-z(2-z)p^2)}{[k^2+z(1-z)p^2]^3} \\
&= e^3(1-\xi)\int \frac{d^Dk}{(2\pi)^D}2\int_0^1 dz(1-z)\frac{-zp^\mu k^2+k^\mu 2(1-z)k\cdot p}{[k^2]^3}+\text{finite} \\
&= e^3(1-\xi)\int \frac{d^Dk}{(2\pi)^D}2\int_0^1 dz(1-z)\frac{-zp^\mu+\frac{1}{2}(1-z)p^\mu}{[k^2]^2}+\text{finite} \\
&= e^3(1-\xi)2\int_0^1 dz(1-z)\frac{1}{2}(1-3z)p^\mu \frac{i}{(4\pi)^2}\Gamma(\epsilon)+\text{finite} \\
&= 0+\text{finite}. \qquad (57)
\end{aligned}
$$

The third diagram gives the same contribution as the second one except for the replacement $p \to p'$. Overall, the vertex correction is

$$
e^3(p+p')^\mu \frac{i}{(4\pi)^2}\Gamma(\epsilon)[-1+(1-\xi)+3] = ie(p+p')^\mu \frac{(3-\xi)e^2}{(4\pi)^2}\Gamma(\epsilon). \qquad (58)
$$

Together with the tree-level piece,

$$
\Gamma^{(3)} = -ie(p+p')^\mu \left[1-\frac{(3-\xi)e^2}{(4\pi)^2}\Gamma(\epsilon)\right] = -ie(p+p')^\mu Z_1^{-1}. \qquad (59)
$$

Therefore,

$$
Z_1 = 1+\frac{(3-\xi)e^2}{(4\pi)^2}\Gamma(\epsilon)M^{-2\epsilon}+\text{finite}. \qquad (60)
$$

Note that this is exactly the same as the divergent piece in $Z_2$. We could of course also verify the finite pieces are the same, but it is beyond the scope of this solution set. This way, we have verified that our calculation of $\beta_e$ using the vacuum polarization diagrams alone is correct.

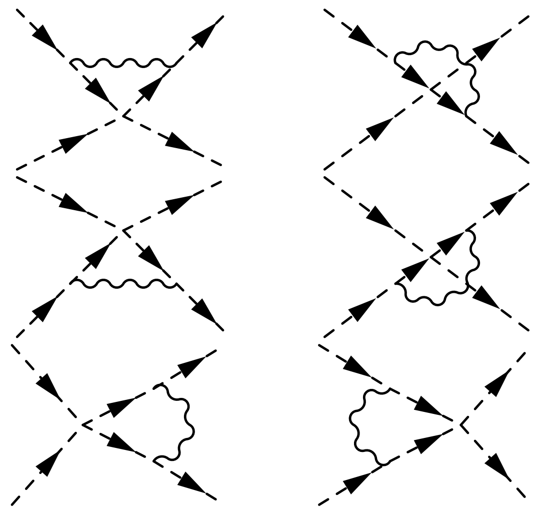

Figure 6: The 1PI four-point function of scalars at $O(\lambda e^2)$.

### (e.4.3) $O(e^2\lambda)$ Vertex Correction

There are 1PI four-point diagrams Fig. 6 that renormalize $\lambda$ at $O(e^2\lambda)$. The first diagram is

$$\frac{-i2\lambda}{3}(-ie)^2\int\frac{d^Dk}{(2\pi)^D}\frac{i}{(p+k)^2}\frac{i}{(p'+k)^2}(2p+k)^\mu(2p'+k)^\nu\frac{-i(g_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2})}{k^2}. \tag{61}$$

We extract only the ultraviolet-divergent pole piece, and for this purpose we can set $p=p'=0$. To avoid infrared divergence, we can stick in a fictitious small mass $m^2$ to all propagators. Then,

$$-\frac{2e^2\lambda}{3}\int\frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-m^2}\frac{1}{k^2-m^2}(1-(1-\xi)) = -\frac{2e^2\lambda}{3}\frac{i}{(4\pi)^2}\Gamma(\epsilon)\xi+\text{regular}. \tag{62}$$

Note that the first four diagrams all give identical divergent piece, while the last two have the opposite sign because of the direction of arrows. Therefore, the overall contribution of six diagrams is

$$-i\frac{2\lambda}{3}\frac{2e^2}{(4\pi)^2}\Gamma(\epsilon)\xi+\text{regular}. \tag{63}$$

This is added to the tree-level amplitude $-i\frac{2\lambda}{3}$, while the scattering amplitude comes with the wave function renormalization factors for all four external

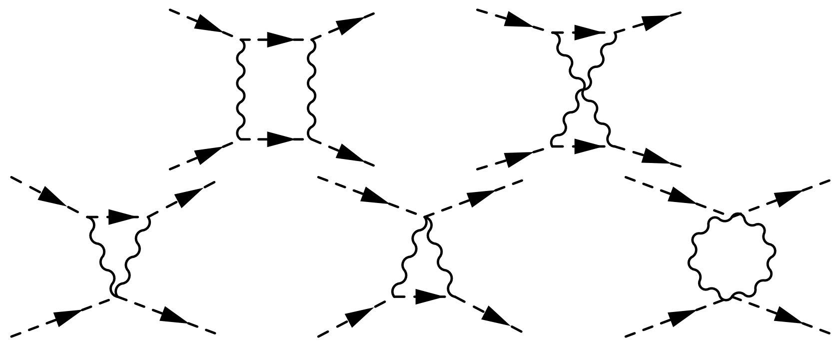

Figure 7: The 1PI four-point function of scalars at $O(e^4)$. For each diagram, there is an additional diagram where the two outgoing lines are interchanged.

lines. Therefore the overall amplitude at $O(e^2\lambda)$ is given by

$$
\begin{aligned}
&-i\frac{2\lambda}{3}\left[1+\frac{2e^2}{(4\pi)^2}\Gamma(\epsilon)\xi\right](\sqrt{Z})^4 \\
&\quad= -i\frac{2\lambda}{3}\left[1+\frac{2e^2}{(4\pi)^2}\Gamma(\epsilon)\xi\right]\left[1-\frac{e^2}{(4\pi)^2}\Gamma(\epsilon)(-3+\xi)\right]^2 \\
&\quad= -i\frac{2\lambda}{3}\left[1+\frac{6e^2}{(4\pi)^2}\Gamma(\epsilon)\right]. \qquad (64)
\end{aligned}
$$

Recovering $M$ dependence , the renormalized $\lambda$ depends on $M$ as

$$
M\frac{d\lambda}{dM} = M\frac{d}{dM}\lambda\left[1+\frac{6e^2}{(4\pi)^2}\Gamma(\epsilon)M^{-2\epsilon}\right] = -\frac{3e^2\lambda}{4\pi^2}\ . \qquad (65)
$$

This is indeed the $O(e^2\lambda)$ contribution to $\beta_\lambda$ in Eq. (52) we obtained from the effective potential and Callan–Symanzik equation.

### (e.4.4) $O(e^4)$ Vertex Correction

The $O(e^4)$ contribution to $\beta_\lambda$ comes from the diagrams in Fig. 7. Again we take external momenta to zero to pick up the UV divergent piece. The IR divergence can be regulated by a finite mass, which we don't bother to write any more. The first diagram is

$$
(-ie)^4\int\frac{d^Dk}{(2\pi)^D}
$$

$$k^\mu \frac{i}{k^2} k^\nu \frac{-i(g_{\nu\rho} - (1-\xi)\frac{k_\nu k_\rho}{k^2})}{k^2} (-k)^\rho \frac{i}{k^2} (-k)^\sigma \frac{-i(g_{\mu\sigma} - (1-\xi)\frac{k_\mu k_\sigma}{k^2})}{k^2}$$
$$= e^4 \xi^2 \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 k^2} = \frac{i}{(4\pi)^2} e^4 \xi^2 \Gamma(\epsilon) + \text{regular}. \quad (66)$$

The second diagram gives the same contribution to the divergent piece. The third diagram is

$$(2ie^2 g_{\mu\nu})(-ie)^2 \int \frac{d^D k}{(2\pi)^D}$$
$$k^\rho \frac{i}{k^2} k^\sigma \frac{-i(g_{\nu\rho} - (1-\xi)\frac{k_\nu k_\rho}{k^2})}{k^2} \frac{-i(g_{\mu\sigma} - (1-\xi)\frac{k_\mu k_\sigma}{k^2})}{k^2}$$
$$= -2e^4 \xi^2 \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 k^2} = -\frac{i}{(4\pi)^2} 2e^4 \xi^2 \Gamma(\epsilon) + \text{regular}. \quad (67)$$

The fourth diagram gives the same contribution to the divergent piece. Finally, the fifth diagram is

$$\frac{1}{2}(2ie^2 g_{\mu\nu})(2ie^2 g_{\rho\sigma}) \int \frac{d^D k}{(2\pi)^D} \frac{-i(g_{\nu\rho} - (1-\xi)\frac{k_\nu k_\rho}{k^2})}{k^2} \frac{-i(g_{\mu\sigma} - (1-\xi)\frac{k_\mu k_\sigma}{k^2})}{k^2} . \quad (68)$$

The overall factor of $\frac{1}{2}$ is there because of the closed loop of identical bosons (photon). The UV-divergent contribution is

$$2e^4 \int \frac{d^D k}{(2\pi)^D} \frac{3+\xi^2}{k^2 k^2} = \frac{i}{(4\pi)^2} 2e^4 (3+\xi^2) \Gamma(\epsilon) + \text{regular}. \quad (69)$$

Because the two final-state scalars can be interchanged, each of the contribution above is further doubled. Summing all five diagrams with a factor of two, we find

$$2\frac{i}{(4\pi)^2} \left[2e^4\xi^2 - 4e^4\xi^2 + 2e^4(3+\xi^2)\right] \Gamma(\epsilon) = \frac{i}{(4\pi)^2} 12 e^4 \Gamma(\epsilon). \quad (70)$$

Because this contribution adds to $-i2\lambda/3$, the $O(e^4)$ contribution to $\beta_\lambda$ is

$$M\frac{d\lambda}{dM} = -M\frac{d}{dM}\frac{3}{2}\frac{1}{(4\pi)^2} 12 e^4 \Gamma(\epsilon) M^{-2\epsilon} = \frac{36}{(4\pi)^2} e^4 = \frac{54 e^4}{24\pi^2} , \quad (71)$$

which agrees with what we obtained from the effective potential and the Callan–Symanzik equation Eq. (52).

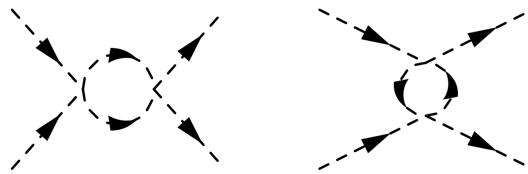

Figure 8: The 1PI four-point function of scalars at $O(\lambda^2)$. The first $s$-chanel diagram has two identical bosons running inside the loop, and hence has a factor of 1/2. The second $t$-channel diagram comes with the crossed $u$-channel diagram as well.

### (e.4.5) $O(\lambda^2)$ vertex correction

Finally, we compute the easiest piece, $O(\lambda^2)$ contribution to $\beta_\lambda$. The only tricky issue is the combinatoric factors. The $s$-channel diagram has two identical bosons running inside the loop and hence comes with a factor of 1/2, while the $t$- and $u$-channel diagrams don't. Therefore the overall divergent piece is

$$\left(-i\frac{2\lambda}{3}\right)^2 \int \frac{d^Dk}{(2\pi)^D}\frac{i}{k^2}\frac{i}{k^2}\left(\frac{1}{2}+1+1\right) = \frac{i}{(4\pi)^2}\left(\frac{2\lambda}{3}\right)^2\frac{5}{2}\Gamma(\epsilon). \tag{72}$$

Since this is the correction to $-i\frac{2\lambda}{3}$, the contribution to $\beta_\lambda$ at $O(\lambda^2)$ is

$$\beta_\lambda = M\frac{d}{dM}\frac{1}{(4\pi)^2}\frac{-2\lambda^2}{3}\frac{5}{2}\frac{1}{\epsilon}M^{-2\epsilon} = \frac{\lambda^2}{(4\pi)^2}\frac{10}{3} = \frac{5\lambda^2}{24\pi^2}\ , \tag{73}$$

which agrees with Eq. (52).

## (e.5) Renormalization-group Flow

We rewrite the evolution equations for $\epsilon = e^2/24\pi^2$ and $l = \lambda/24\pi^2$, and $t = -\ln M$,

$$\frac{d}{dt}\epsilon = -\epsilon^2 \tag{74}$$

$$\frac{d}{dt}l = -(5l^2 - 18l\epsilon + 54\epsilon^2). \tag{75}$$

The important point is that both equations have negative definite l.h.s. and hence both of the couplings go to zero in the infrared limit. Note $5l^2 - 18l\epsilon +$

$54\epsilon^2 = 5(l - \frac{9}{5}\epsilon)^2 + \frac{189}{5}\epsilon^2 > 0$. The problem is asking us to see the relative size between the two.

First of all, the running of $l$ is much faster than that of $\epsilon$ becase of the larger l.h.s. Therefore, even if the initial condition has $l > \epsilon$, $l$ quickly becomes smaller than $\epsilon$. Therefore, without a loss of generality, we can assume $l < \epsilon$.

The first equation can be solved analytically and we find

$$\frac{1}{\epsilon(t)} - \frac{1}{\epsilon(0)} = t \tag{76}$$

and hence

$$\epsilon(t) = \frac{\epsilon(0)}{1 + \epsilon(0)t} . \tag{77}$$

Therefore,

$$e^2(t) = \frac{1}{e^{-2}(0) + \frac{1}{24\pi^2}t} . \tag{78}$$

Assuming $l$ is negligible already compared to $\epsilon$ in the second equation, we need to solve

$$\frac{d}{dt}l = -\frac{54\epsilon(0)^2}{(1 + \epsilon(0)t)^2} . \tag{79}$$

The solution is found easily as

$$l(t) - l(0) = \frac{54\epsilon(0)}{1 + \epsilon(0)t} - 54\epsilon(0) = -\frac{54\epsilon(0)^2 t}{1 + \epsilon(0)t} . \tag{80}$$

By assumption, for some $t_0$, $l(t_0) < \epsilon(t_0) = 1/t_0$. Therefore, the terms in the parentheses are negative. What it means is that $l$ crosses zero at some $t$ and goes negative evententually.

This behavior justifies the assumption in part (c) that $0 < \lambda \approx e^4 \ll e^2$ just before it crosses zero.