Functional Method

We solve Problem 9.2 in Peskin–Schroeder.

(a)

We would like to evaluate the partition function

$$Z = \mathrm{tr}[e^{-\beta H}] \tag{1}$$

using the path integral. First of all, the trace of an arbitrary operator $\mathcal{O}$ over the Hilbert space can be taken in any basis, *e.g.*, energy eigenstates $|n\rangle$ or position eigenstates $|x\rangle$,

$$\begin{aligned}
\mathrm{tr}\mathcal{O} &= \sum_n \langle n|\mathcal{O}|n\rangle \\
&= \int\int \sum_n \langle n|x\rangle dx \langle x|\mathcal{O}|y\rangle dy \langle y|n\rangle \\
&= \int\int dxdy \langle x|\mathcal{O}|y\rangle \sum_n \langle y|n\rangle\langle n|x\rangle \\
&= \int\int dxdy \langle x|\mathcal{O}|y\rangle \langle y|x\rangle \\
&= \int\int dxdy \langle x|\mathcal{O}|y\rangle \delta(x-y) \\
&= \int dx \langle x|\mathcal{O}|x\rangle.
\end{aligned} \tag{2}$$

The next step is to divide up the Boltzmann factor $e^{-\beta H}$ into many many small pieces,

$$\begin{aligned}
\mathrm{tr}[e^{-\beta H}] &= \int dx \langle x|e^{-\beta H}|x\rangle \\
&= \int dx \prod_{i=1}^{N-1} \langle x|e^{-\beta H/N}|x_{N-1}\rangle dx_{N-1} \langle x_{N-1}|e^{-\beta H/N}|x_{N-2}\rangle dx_{N-2} \langle x_{N-2}| \cdots \\
&\quad \cdots |x_1\rangle dx_1 \langle x_1|e^{-\beta H/N}|x\rangle
\end{aligned} \tag{3}$$

For the Hamiltonian $H = \frac{p^2}{2m} + V(x)$, we compute the matrix element $\langle x|e^{-\epsilon H}|y\rangle$ for small $\epsilon = \beta/N$. We find

$$\langle x|e^{-\epsilon H}|y\rangle = \langle x|e^{-\epsilon p^2/2m}e^{-\epsilon V(x)}e^{-O(\epsilon^2)}|y\rangle \tag{4}$$

because of the Baker–Campbell–Hausdorff formula. We ignore $O(\epsilon^2)$ piece in the exponent. Then we insert the complete set of momentum eigenstates,

$$\begin{aligned} \langle x|e^{-\epsilon H}|y\rangle &= \int \langle x|e^{-\epsilon p^2/2m}|p\rangle dp \langle p|e^{-\epsilon V(x)}e^{-O(\epsilon^2)}|y\rangle \\ &= \int dp \frac{1}{\sqrt{2\pi\hbar}} e^{-\epsilon p^2/2m} e^{ipx/\hbar} \frac{1}{\sqrt{2\pi\hbar}} e^{-\epsilon V(y)} e^{-ipy/\hbar} \\ &= \frac{1}{2\pi\hbar}\sqrt{\frac{2\pi m}{\epsilon}} e^{-m(x-y)^2/2\epsilon\hbar^2} e^{-\epsilon V(y)} \end{aligned} \tag{5}$$

Therefore, using the notation $x_0 = x_N = x$,

$$\mathrm{tr}[e^{-\beta H}] = \int \prod_{i=1}^{N} \frac{dx_i}{\sqrt{2\pi\hbar^2\epsilon}} e^{-S_E/\hbar}, \tag{6}$$

where

$$\begin{aligned} S_E &= \hbar \sum_{i=1}^{N} \left[\frac{m(x_i - x_{i-1})^2}{2\epsilon\hbar^2} + \epsilon V(x_i)\right] \\ &= \frac{\hbar\beta}{N} \sum_{i=1}^{N} \left[\frac{m(x_i - x_{i-1})^2}{2(\hbar\beta/N)^2} + V(x_i)\right] \\ &\to \oint_0^{\hbar\beta} \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right] d\tau. \end{aligned} \tag{7}$$

At the last step, the limit $N \to \infty$ was taken. This is nothing but the Euclidean action, namely the action after the Wick rotation $t = -i\tau$. The path $x(\tau)$ satisfies the periodic boundary condition $x(\hbar\beta) = x(0)$.

(b)

The specified expansion ($\hbar = 1$ below),[1]

$$x = \frac{1}{\sqrt{\beta}} \sum_n x_n e^{2\pi i n\tau/\beta} \tag{8}$$

[1] I use τ instead of t to distinguish real and imaginary times.

implies that $x_{-n} = x_n^*$. In particular, x_0 is real. The Euclidean action is

$$\begin{aligned} S_E &= \oint_0^\beta d\tau \sum_{n,m} \frac{1}{2}\left[\frac{2\pi i n}{\beta}\frac{2\pi i m}{\beta} + \omega^2\right]\frac{1}{\beta} x_n e^{2\pi i n\tau/\beta} x_m e^{2\pi i m\tau/\beta} \\ &= \frac{1}{2}\omega^2 x_0^2 + \sum_{n=1}^\infty \left(\frac{(2\pi n)^2}{\beta^2} + \omega^2\right)|x_n|^2. \end{aligned} \tag{9}$$

In the second line, we used $m = -n$ from the τ integral. Therefore, the path integral is Gaussian and is given by the product of eigenvalues up to an overall β-dependent (but ω-independent) factor,

$$\begin{aligned} Z &\propto \left[\omega \prod_{n=1}^\infty \left(\frac{(2\pi n)^2}{\beta^2} + \omega^2\right)\right]^{-1} \\ &\propto \left[\frac{\beta\omega}{2} \prod_{n=1}^\infty \left(1 + \frac{\beta^2\omega^2}{(2\pi n)^2}\right)\right]^{-1} = \left[\sinh\frac{\beta\omega}{2}\right]^{-1}. \end{aligned} \tag{10}$$

Here, we used the infinite product representation of $\sinh z$. Up to a β-dependent (but ω-independent) factor, it is[2]

$$Z \propto \left[e^{\beta\omega/2} - e^{-\beta\omega/2}\right]^{-1} = \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}}. \tag{11}$$

This precisely matches the known expression of the partition function for a harmonic oscillator,

$$Z = \sum_{n=0}^\infty e^{-\beta\omega(n+1/2)} = \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}}. \tag{12}$$

As a prepration to the next problem, it is instructive to rewrite the Euclidean action as

$$S_E = \oint_0^{\hbar\beta} d\tau\, \frac{1}{2} x(\tau)\left[-\partial_\tau^2 + \omega^2\right] x(\tau), \tag{13}$$

and hence

$$Z = [\det(-\partial_\tau^2 + \omega^2)]^{-1/2} = \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}}. \tag{14}$$

[2] One can of course work out the overall factor carefully with a lot more work. See `http://hitoshi.berkeley.edu/221A/pathintegral.pdf`

(c)

The partition function is given in terms of the path integral

$$Z = \int \mathcal{D}\phi(\vec{x}, \tau) e^{-S_E}, \tag{15}$$

where the Euclidean action is ($\hbar = c = 1$)

$$S_E = \int d\vec{x} \oint_0^\beta d\tau \, \frac{1}{2}\left[\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right] = \int d\vec{x} \oint_0^\beta d\tau \, \frac{1}{2}\phi(-\partial^2 + m^2)\phi. \tag{16}$$

Here, $\dot{\phi} = d\phi/d\tau$. The path integral therefore yields

$$Z = [\det(-\partial^2 + m^2)]^{-1/2} = [\det(-\partial_\tau^2 - \vec{\nabla}^2 + m^2)]^{-1/2}. \tag{17}$$

This determinant is a product of many momentum modes $\vec{\nabla} = i\vec{p}$,

$$Z = \prod_{\vec{p}} [\det(-\partial_\tau^2 + \vec{p}^2 + m^2)]^{-1/2}. \tag{18}$$

Using the result from the part (b), the determinant is

$$Z = \prod_{\vec{p}} \frac{e^{-\beta\omega(\vec{p})/2}}{1 - e^{-\beta\omega(\vec{p})}}, \tag{19}$$

where $\omega(\vec{p}) = \sqrt{\vec{p}^2 + m^2}$. This expression is indeed the partition function for a relativistic boson of mass m.

(d)

We are given the Euclidean action

$$S_E = \oint_0^\beta d\tau \left[\bar{\psi}\dot{\psi} + \omega\bar{\psi}\psi\right], \tag{20}$$

Because of the anti-periodic boundary condition $\psi(\beta) = -\psi(0)$, we expand it in Fourier series

$$\psi(\tau) = \frac{1}{\sqrt{\beta}} \sum_n \psi_n e^{2\pi i(n+\frac{1}{2})\tau/\beta}, \qquad \bar{\psi}(\tau) = \frac{1}{\sqrt{\beta}} \sum_n \bar{\psi}_n e^{-2\pi i(n+\frac{1}{2})\tau/\beta}. \tag{21}$$

Then the Euclidean action is

$$\begin{aligned} S_E &= \oint_0^\beta d\tau \frac{1}{\beta} \sum_{n,m} \bar{\psi}_n e^{-2\pi i(n+\frac{1}{2})\tau/\beta} \left(\frac{2\pi i}{\beta} \left(m + \frac{1}{2} \right) + \omega \right) \psi_m e^{2\pi i(m+\frac{1}{2})\tau/\beta} \\ &= \sum_n \bar{\psi}_n \left(\frac{2\pi i}{\beta} \left(n + \frac{1}{2} \right) + \omega \right) \psi_n. \end{aligned} \tag{22}$$

Therefore the path integral over Grasmannian variables gives

$$Z = \int \prod_n d\psi_n d\bar{\psi}_n e^{-S_E} = \prod_n \left[\frac{2\pi i}{\beta} \left(n + \frac{1}{2} \right) + \omega \right] \tag{23}$$

This infinite product can be rewritten as

$$\begin{aligned} Z &= \prod_{n=0}^{\infty} \left[\frac{2\pi i}{\beta} \left(n + \frac{1}{2} \right) + \omega \right] \left[-\frac{2\pi i}{\beta} \left(n + \frac{1}{2} \right) + \omega \right] \\ &= \prod_{n=0}^{\infty} \left[\left(\frac{2\pi}{\beta} \right)^2 \left(n + \frac{1}{2} \right)^2 + \omega^2 \right]. \end{aligned} \tag{24}$$

It is useful to recall the infinite product representation of cosh,

$$\cosh x = \prod_{n=0}^{\infty} \left[1 + \frac{x^2}{\pi^2 (n + \frac{1}{2})^2} \right]. \tag{25}$$

We find

$$\begin{aligned} Z &= \prod_{n=0}^{\infty} \left(\frac{2\pi}{\beta} \right)^2 \left(n + \frac{1}{2} \right)^2 \left[1 + \left(\frac{\beta\omega}{2\pi(n + \frac{1}{2})} \right)^2 \right] \\ &\propto \cosh \frac{\beta\omega}{2} \propto e^{\beta\omega/2} + e^{-\beta\omega/2}. \end{aligned} \tag{26}$$

On the other hand, the canonical quantization of the same Lagrangian in the real time is based on the anti-commutation relations $\{\psi, \bar{\psi}\} = 1$ and $\{\psi, \psi\} = \{\bar{\psi}, \bar{\psi}\} = 0$ with the Hamiltonian $H = \omega \frac{1}{2} (\bar{\psi}\psi - \psi\bar{\psi}) = \omega \left(\bar{\psi}\psi - \frac{1}{2} \right)$. The ground state is annihilated by the annihilation operator $\psi|0\rangle = 0$ while the excited state is $|1\rangle = \bar{\psi}|0\rangle$. Because of the anticommutation $\{\bar{\psi}, \bar{\psi}\} = 2\bar{\psi}^2 = 0$, one cannot occupy the sate with more than once, $\bar{\psi}|1\rangle = \bar{\psi}^2|0\rangle = 0$. The energy eigenvalues are

$$H = \omega \left(\bar{\psi}\psi - \frac{1}{2} \right) = \pm \frac{\omega}{2}. \tag{27}$$

Therefore, the partition function is

$$Z = e^{\beta\omega/2} + e^{-\beta\omega/2} = 2\cosh\frac{\beta\omega}{2}, \tag{28}$$

which agrees with the path integral up to an overall factor.

(e)

The Euclidean path integral including the gauge fixing as in Eq. (9.56) (Peskin–Schroeder) is

$$Z = \int \mathcal{D}A_\mu e^{-\int d^4x\left(\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\xi}(\partial_\mu A_\mu)^2\right)}(\det\,\partial^2) \tag{29}$$

up to an overall constant. In Feynman gauge ($\xi = 1$), the calculation is extremely simple.

The Euclidean action can be rewritten as

$$\begin{aligned} S_E &= \int d^4x \left(\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2}(\partial_\mu A_\mu)^2\right) \\ &= \int d^4x \frac{1}{2}\left[\partial_\mu A_\nu(\partial_\mu A_\nu - \partial_\nu A_\mu) + (\partial_\nu A_\nu)(\partial_\mu A_\mu)\right] \\ &= \frac{1}{2}\int d^4x \left[-A_\nu(\partial^2 A_\nu - \partial_\nu(\partial_\mu A_\mu)) - A_\nu\partial_\nu(\partial_\mu A_\mu)\right] \\ &= -\frac{1}{2}\int d^4x A_\nu \partial^2 A_\nu. \end{aligned} \tag{30}$$

Therefore, it is nothing but a collection of four scalar fields ($\mu = 0, 1, 2, 3$), and its integral over A_μ yields $[(\det\,\partial^2)^{-1/2}]^4 = (\det\,\partial^2)^{-2}$. On the other hand, the Faddeev–Popov determinant ($\det\,\partial^2$) cancels one of the powers and hence

$$Z = (\det\,\partial^2)^{-1}. \tag{31}$$

This is square of the determinant of a massless boson, correctly account for two polarization states of the photon for the partition function.