Lorentz-covariant spectrum of single-particle states and their field theory

Physics 230A, Spring 2007, Hitoshi Murayama

1 Poincaré Symmetry

In order to understand the number of degrees of freedom we need to include in a Lorentz-invariant theory, we need to develop the representation theory of the Poincaré symmetry. The Poincaré symmetry consists of two sets of symmetries of the Minkowski spacetime. Space-time translations generated by the energy-momentum four vector P^μ, and the Lorentz transformations (rotation and boost) generated by the operators $M^{\mu\nu}$.

I can't remember the commutation relations, and I always work them out using the analogy to the single-particle quantum mechanics. The translation generates are energy and momentum $P_\mu = i\partial_\mu$, and the boost and rotation $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$.* The commutators are

$$[P^\mu, P^\nu] = 0, \tag{1}$$

for the translation generators,

$$\begin{aligned} [M^{\mu\nu}, M^{\rho\sigma}] &= [x^\mu i\partial^\nu - x^\nu i\partial^\mu, x^\rho i\partial^\sigma - x^\sigma i\partial^\rho] \\ &= ig^{\nu\rho} M^{\mu\sigma} - ig^{\mu\rho} M^{\nu\sigma} - ig^{\nu\sigma} M^{\mu\rho} - ig^{\mu\sigma} M^{\nu\rho}, \end{aligned} \tag{2}$$

for the boost and rotation generators, and

$$[M^{\mu\nu}, P^\rho] = [x^\mu i\partial^\nu - x^\nu i\partial^\mu, i\partial^\rho] = g^{\mu\rho}\partial^\nu - g^{\nu\rho}\partial^\mu = -i(g^{\mu\rho}P^\nu - g^{\nu\rho}P^\mu), \tag{3}$$

between them. These expressions are true for any dimensions, even for different signatures, such as Euclidean space.

If you specialize them to the spatial components only in three-dimensional space, we recover familiar commutators from quantum mechanics, by identifying $J^k = \epsilon^{klm} M^{lm}$,

$$[J^1, J^2] = [M^{23}, M^{31}] = ig^{33} M^{21} = i(-1)(-M^{12}) = iJ^3, \tag{4}$$

$$[J^1, P^2] = [M^{23}, P^2] = -ig^{22} P^3 = iP^3. \tag{5}$$

*Peskin–Schroeder uses the notation $J^{\mu\nu}$ in Chapter 3. Somehow I can't get used to this notation and I stick with mine: $M^{\mu\nu}$.

In four dimensions, there are two Casimir operators of the Poincaré symmetry. One is $P^2 = P^\mu P_\mu$, which obviously commutes with P^μ and is also Lorentz invariant and hence commutes with $M^{\mu\nu}$. The other is made of Pauli-Lubanski pseudo-vector

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma}. \tag{6}$$

Note that the orbital angular momentum drops out from $M_{\rho\sigma}$ because of the anti-symmetry with the momentum vector. Namely it picks up only the spin part of the angular momentum.

Commutators of W^μ with P^κ leave another P_ρ or P_σ contracted with the Levi–Civita symbol and hence vanish. It transforms as a Lorentz vector and hence has the same commutator with $M^{\mu\nu}$ as the momentum vector, but has the opposite transformation under parity. Clearly $W^2 = W^\mu W_\mu$ is Lorentz invariant and hence commutes with all generators.

In general in even dimensions $D = 2k$, one can define higher dimensional analog of the Pauli–Lubanski pseudo-tensors of the type $PM, PM^2, \cdots, PM^{k-1}$, and their Lorentz-invariant squares. Together with P^2, there are k Casimir operators.

In odd dimensions $D = 2k + 1$, there are the same k Casimir operators and an additional one $\epsilon^{\mu_1\mu_2\cdots\mu_{2k+1}} P_{\mu_1} M_{\mu_2\mu_3} \cdots M_{\mu_{2k}\mu_{2k+1}}$ that does not need to be squared to be Lorentz-invariant.

In order to understand representations of the Poincaré group, we first note that it is a non-compact group because the Lorentz boost can be continued indefinitely, and also space and time translations. A non-compact group in general does not admit a finite-dimensional unitarity representation, except for when non-compact generators can be consistently be set to zero (then it reduces to the representation theory of a compact group). Therefore, the Poincaré group with non-zero four-momentum has only infinite dimensional representations on Hilbert space (which by definition requires unitarity). In other words, there are infinite number of states with different four-momenta $P^\mu = m\gamma(1, \beta \sin\theta \cos\phi, \beta \sin\theta \sin\phi, \beta \cos\theta)$ related by Lorentz boosts. What concerns us is the multiplicity of states (the number of degrees of freedom) for a particle of fixed four momentum. That is the issue of the little group in the next section.

2 Little Group

In order to develop the representation theory of the Poincaré algebra, we first look at the Casimir operator P^2. Clearly, physics is quite different for $P^2 > 0$ (massive), $P^2 = 0$ (massless), and $P^2 < 0$ (tachyonic). To understand the multiplicity of the single particle state, we choose a particular four-momentum in each case and work out the representation theory of remaining symmetries. The remaining symmetry that keeps the four-momentum unchanged is called Wigner's little group.

(a) Massive case

This is the case when $P^2 = m^2 > 0$. We can choose the rest frame, namely the reference frame in which $P^\mu = (m, 0, \cdots, 0)$, without a loss of generality by using a Lorentz boost.

The remaining generators that leave this four-momentum unchanged is M^{ij} where $i, j = 1, \cdots D-1$, namely spatial rotations without boosts. They form the little group $SO(D-1)$, whose representation theory is well understood. In the four-dimensional spacetime, it is $SO(3)$, whose Lie algebra is the same as that of $SU(2)$. The representations are given in terms of the spin j with the multiplicity $2j+1$.

In general, a massive spin 1 particle is a vector representation of the little group $SO(D-1)$ and hence has $D-1$ degrees of freedom. A massive spin 2 particle is a traceless symmetric rank-two tensor and has $D(D-1)/2-1$ independent components. A massive spin 1/2 particle has $2^{[(D-2)/2]}$ degrees of freedom, where [.] is the Gauss' symbol to take the largest integer equal or less than the argument.[†]

In the four-dimensional case, the Pauli–Lubanski vector is

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} P_\nu M_{\rho\sigma} = \frac{1}{2} m \epsilon^{\mu 0 \rho\sigma} M_{\rho\sigma}. \tag{7}$$

Therefore,

$$W^0 = 0, \qquad W^i = -m s^i \tag{8}$$

where $\vec{s}$ is the spin operator. The Casimir $W^\mu W_\mu = -m^2 \vec{s}^2 = -m^2 s(s+1)$. $\vec{s}^2$ is the Casimir operator of $SO(3)$ rotation group.

[†]For the construction of spin 1/2 representations in general dimensions, see the lecture notes on Clifford algebra.

In general, the Casimir operators of the Poincaré algebra (apart from $P^2 = m^2$) correspond to the Casimir operators of the little group $SO(D-1)$; there are k of them for $SO(2k)$ and $SO(2k+1)$.

(b) Massless case

This is quite different from the massive case. Without a loss of generality, we can choose $P^\mu = E(1, 1, 0, \cdots, 0)$. Clearly the generators M^{ij} with $i, j = 2, \cdots D$ leave the four-momentum unchanged. There are, however, other generators that also leave the four-momentum unchanged. For any $i = 2, \cdots, D$, $P^i = 0$ and hence

$$[M^{0i}, P^\mu] = -i(g^{0\mu} P^i - g^{i\mu} P^0) = i g^{i\mu} P^0, \tag{9}$$

$$[M^{1i}, P^\mu] = -i(g^{1\mu} P^i - g^{i\mu} P^1) = i g^{i\mu} P^1. \tag{10}$$

Because $P^0 = P^1$, we find

$$[M^{0i} - M^{1i}, P^\mu] = 0. \tag{11}$$

Therefore, M^{ij} and $K^i \equiv M^{0i} - M^{1i}$ form the little group. The K^i's commute with each other, and transform as a vector under M^{ij},

$$[M^{ij}, K^k] = -i(g^{ik} K^j - g^{jk} K^i) = i(\delta^{ik} K^j - \delta^{jk} K^i). \tag{12}$$

In other words, they form the "Poincaré algebra" of $D-2$ dimensional Euclidean space, which is called the "Euclidean motion group" E_{D-2} which consists of rotations and spatial translations.[‡]

Since the Euclidean motion group is non-compact, in general its unitary representations are infinite dimensional. We do not want a single particle to have infinite degrees of freedom.[§] Then we must set all of the non-compact generators K^i identically zero, which is consistent with the algebra. The remaining generators form a compact group $SO(D-2)$ (called maximum compact subgroup of a non-compact group), which have happily finite-dimensional unitary representations.[¶]

[‡]Unfortunately this notation of E_n has nothing to do with $E_{6,7,8}$ exceptional simple Lie groups. I know it is a confusing notation, but it is used in the literature.

[§]If you have a good reason why we should consider this, let me know! Even string theory has only finite number of degrees of freedom for a given mass.

[¶]The mismatch between the little groups, $SO(D-1)$ for massive and $SO(D-2)$ for massless particles, is the source of strong constraints on theories that incorporate both of them, *i.e.*, string theory. The consistency having both leads to the requirement of *critical dimensions*, namely 26 for bosonic and 10 for supersymmetric string.

A massless spin one particle (gauge boson) has $D-2$ components, while the massless spin two (graviton) $(D-1)(D-2)/2-1$. A massless spin 1/2 particle has $2^{[(D-3)/2]}$ independent components.

Setting $D=4$, we find 2 components for spin 1, again 2 for spin two, and one for spin 1/2. Note for four dimensions, the maximum compact subgroup of the little group is $SO(2)$, which has only one-dimensional irreducible representations. Each irreducible representation corresponds to a definite helicity state. The CPT theorem requires that for each state of helicity h, there must be a CPT conjugate state of helicity $-h$. For spin j, you find the particle state of helicity $+j$ and the anti-particle state of helicity $-j$ (or vice versa). Two states for spins 1 and 2 already have CPT pairs. One state for spin 1/2 must be accompanied with its anti-particle state (Weyl fermion).

The Pauli–Lubanski vector is

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma} = \frac{1}{2}E(\epsilon^{\mu 0\rho\sigma}M_{\rho\sigma} - \epsilon^{\mu 1\rho\sigma}M_{\rho\sigma}). \tag{13}$$

Therefore,

$$W^0 = -EJ^1, \quad W^1 = -EJ^1, \quad W^2 = -EJ^2 - EM^{03}, \quad W^3 = -EJ^3 + EM^{02}. \tag{14}$$

Because $K^i = M^{0i} - M^{1i} = 0$, we find $W^2 = W^3 = 0$, and hence $W^\mu W_\mu = 0$.

(c) Tachyonic case

A particle with negative mass-squared $P^2 = -\mu^2 < 0$ is said to be a tachyon. It should not exist because of the causality. The dispersion relation is $E = \sqrt{\vec{p}^2 - \mu^2}$. Remember the group velocity of a wave is given by

$$\vec{v}_g = \frac{\partial \omega}{\partial \vec{k}} = \frac{\partial E}{\partial \vec{p}} = \frac{\vec{p}}{\sqrt{\vec{p}^2 - \mu^2}} > c. \tag{15}$$

It propagates faster than the speed of light. We don't want it, we don't need it. The rest of the discussion is therefore completely academic.

Nonetheless one can ask a mathematical question if there is any finite-dimensional unitary representation for a tachyon. For this purpose, we again fix the reference frame and take $P^\mu = (0, \cdots, 0, \mu)$. The little group that does not change this momentum is $SO(D-2,1)$ generated by $M^{\mu\nu}$ for $\mu, \nu = 0, \cdots, D-2$. This is a non-compact group and hance unitarity representations are in general infinite-dimensional.

Like in the massless case, we can try to set non-compact generators M^{0i} to zero. The problem here, unlike in the massless case, is that $[M^{0i}, M^{0j}] = -iM^{ij}$ and hence $M^{0i} = 0$ implies also $M^{ij} = 0$. Therefore, all generators vanish identically and the representation is trivial. Namely that a scalar (spinless) state is the only possible finite-dimensional unitarity representation of the little group.

You may complain that you can add just a negative mass-squared to a vector field (Maxwell Lagrangian). The point is that the condition $\partial_\mu A^\mu = 0$ is satisfied for A^0 for our choice of $P^\mu = i\partial^\mu = (0, \cdots, 0, \mu)$. However, the state created by A^0 has the negative metric and hence this is not a unitary representation. This is not something you can "gauge it away" because it is a part of the irreducible representation of the little group.

3 Field Theory

What field you should use to write down a quantum field theory for a particle of given spin relies on finite-dimensional and non-unitary representation of the Poincaré symmetry. This is in stark contrast with infinite-dimensional and unitary representation of the Poincaré symmetry on the Hilbert space. The challenge is to find a suitable realization on fields whose quantization yields the desired representation on the Hilbert space.

First of all, a quantum field (by definition) is a function of spacetime and hence the translation generators act as space-time derivatives

$$P_\mu \phi^a(x) = i\partial_\mu \phi^a(x). \tag{16}$$

Here, $\phi^a(x)$ is a generic field, possibly multi-component. Therefore, the translations are taken into account already, and what remains is just the Lorentz transformations $SO(D-1,1)$. What we look for then is finite-dimensional representations of $SO(D-1,1)$ so that we don't need to deal with an infinite-component field. This is a very different requirement from what we discussed about the Poincaré symmetry on the Hilbert space. This mismatch between the representations on the fields and the Hilbert space is what will force us to consider gauge symmetries later on.

Finite-dimensional representations of $SO(D-1,1)$ are obtained basically by Wick rotations of $SO(D)$ representations. They are not unitary, because the boost generators are represented by the anti-hermitian generators, but linear and finite-dimensional by definition. Therefore, we first discuss representation theory of $SO(4)$ before we get to $SO(3,1)$ which is what we are really after.[‖]

The $SO(4)$ group has six generators M_{ij}, $i,j=1,2,3,4$. The Wick rotation from the Minkowski space with the metric $g_{\mu\nu}=\text{diag}(+1,-1,-1,-1)$ by $x^0=ix^4$ gives us $g_{ij}=-\delta_{ij}$. One unique aspect of $SO(4)$ is that its Lie algebra decouples into two pieces, namely $SO(4)=SU(2)\times SU(2)$.[**] Let us work them out explicitly.

We know three of the generators very well: $J_x=M_{23}$, $J_y=M_{31}$, $J_z=M_{12}$ with their usual commutation relations. Let us define $K_i=M_{i4}$, where $i=1,2,3$. Using the commutation relations Eq. (2) with $g_{\mu\nu}=-\delta_{\mu\nu}$ in the Euclidean space, it is easy to verify

$$[J_k,J_l]=i\epsilon_{klm}J_m,\qquad [J_k,K_l]=i\epsilon_{klm}K_m,\qquad [K_k,K_l]=i\epsilon_{klm}J_m. \tag{17}$$

Now we define new generators $J_i^+=(J_i+K_i)/2$ which satisfy

$$[J_k^+,J_l^+]=\frac{1}{4}[J_k+K_k,J_l+K_l]=\frac{1}{4}i\epsilon_{klm}(J_m+K_m+K_m+J_m)=i\epsilon_{klm}J_m^+. \tag{18}$$

Similarly for $J_i^-=(J_i-K_i)/2$,

$$[J_k^-,J_l^-]=\frac{1}{4}[J_k-K_k,J_l-K_l]=\frac{1}{4}i\epsilon_{klm}(J_m-K_m-K_m+J_m)=i\epsilon_{klm}J_m^-. \tag{19}$$

We can also check that J_i^+ and J_i^- commute,

$$[J_k^+,J_l^-]=\frac{1}{4}[J_k+K_k,J_l-K_l]=\frac{1}{4}i\epsilon_{klm}(J_m-K_m+K_m-J_m)=0. \tag{20}$$

This way, we have verified that the Lie algebra of $SO(4)$ is nothing but two commuting sets of $SU(2)$. Therefore, the representations are given simply by assigning two "spins" (j_1,j_2) for each $SU(2)$ factors and the dimension of the representation space is $(2j_1+1)(2j_2+1)$. In any of these representations, J_k^+ and J_k^+ are represented as hermitean matrices and so are J_k and K_k.

[‖] The discussion in this section is essentially a solution to Problem 3.1 in the book. Again notations are different; I'm sorry!

[**] Strictly speaking, $SO(4)=(SU(2)\times SU(2))/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts as the diagonal subgroup of the $\mathbb{Z}_2$ centers of each $SU(2)$ factors. This subtlety with global properties of the group does not affect representation theory of the Lie algebra, which relies only on local information around the origin. In fact, the spinor representations are those of $Spin(4)=SU(2)\times SU(2)$, the double cover of $SO(4)$.

The representations of $SO(3,1)$ are obtained by the simple Wick rotation $x^4 = -ix^0$. This gives us the identification $K_i = M_{i4} = -iM_{i0} = iM_{0i}$. Even though M_{0i} are symmetry generators, they are now represented by the anti-hermitean matrices $-iK_i$.

Obviously the smallest representations are $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. They are the spin 1/2 representations for right-handed and left-handed chiralities. For the representation $(\frac{1}{2}, 0)$, $J_i^- = 0$, and hence $K_i = J_i$. On the other hands, $J_i^+ = (J_i + K_i)/2 = \sigma_i/2$, where σ_i are Pauli matrices. In other words, the generators are represented by

$$M_{ij} = \epsilon_{ijk} J_k = \frac{\sigma_k}{2}, \qquad M_{0i} = -iK_i = -i\frac{\sigma_i}{2}. \tag{21}$$

For the other representation $(0, \frac{1}{2})$, we flip the signs of K_i and hence $K_i = -J_i$,

$$J_i = \frac{\sigma_i}{2}, \qquad M_{0i} = -iK_i = +i\frac{\sigma_i}{2}. \tag{22}$$

and hence it is nothing but the hermitean conjugate of the representation $(\frac{1}{2}, 0)$. This is not a paradox. For compact Lie groups the representations are unitary and hence the generators are hermitean. On the other hand for non-compact groups, the generators are not represented by hermitean matrices (and hence the representations are not unitary) and different representations can be related by taking hermitean conjugation.[††]

A rotation matrix is given by

$$R(\vec{\theta}) = e^{i\vec{J}\cdot\vec{\theta}} = e^{i\vec{\sigma}\cdot\vec{\theta}/2} = \cos\frac{\theta}{2} + i\frac{\vec{\sigma}\cdot\vec{\theta}}{\theta}\sin\frac{\theta}{2} \tag{23}$$

as usual. It rotates the system around the axis $\vec{\theta}$ by the angle $\theta = |\vec{\theta}|$. Note that this is common to both $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, and hence we can see how the Lorentz group reduces to the spatial rotation by dropping the distinction between the two.

A Lorentz boost, on the other hand, is given by

$$B(\vec{\eta}) = e^{-iM_{0i}\eta_i} = e^{-i(-i\vec{K})\cdot\vec{\eta}} = e^{\pm\vec{\sigma}\cdot\vec{\eta}/2} = \cosh\frac{\eta}{2} \pm \frac{\vec{\sigma}\cdot\vec{\eta}}{\eta}\sinh\frac{\eta}{2}\ . \tag{24}$$

[††]See Lecture notes on the Clifford algebra for more discussions about how different representations are related to each to other by various kinds of conjugations.

It boosts the system along the direction $\vec{\eta}$ by the rapidity $\eta = \vec{\eta}$. (The rapidity is defined by $\cosh\eta = \gamma = 1/\sqrt{1-\beta^2}$, $\sinh\eta = \gamma\beta$ for Lorentz boosts.) The signs $\pm$ refer to $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ representations, respectively.

The combination of rotations and Lorentz boosts forms the group $SL(2,\mathbb{C})$. A "Special Linear" group is a group of matrices (by definition linear) of unit determinant (special), and this one is defined on complex numbers $\mathbb{C}$. Namely, the Lie algebra of $SO(3,1)$ is equivalent to that of $SL(2,\mathbb{C})$. It is easy to verify that $\det R(\vec{\theta}) = \det B(\vec{\eta}) = 1$ because the exponents are traceless.

In order to describe a spin one particle, we look for representations that contain spin one representation under the rotation group. A Lorentz vector A_μ is given by the $(\frac{1}{2},\frac{1}{2})$ representation. Because the rotation is obtained by dropping the distinction between two $SU(2)$'s, we find $\frac{1}{2}\otimes\frac{1}{2} = 1\oplus 0$. Namely, a spin one component (the spatial components of a four-vector) and a spin zero component (the time component of a four-vector). Therefore it is a candidate for the description of spin one particles, and this is indeed what we use in the gauge theories. The transformation property can be seen more explicitly by using $(\frac{1}{2},0)\otimes(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$. Suppose ξ transforms as $(\frac{1}{2},0)$. Then $\xi^\dagger$ transforms as $(0,\frac{1}{2})$. We form combinations

$$\xi^\dagger\sigma^\mu\xi, \qquad \sigma^\mu = (1,\vec{\sigma}). \tag{25}$$

Under the rotation Eq. (23) the time component does not change,

$$\xi^\dagger\sigma^0\xi \to \xi^\dagger R(\vec{\theta})^\dagger\sigma^0 R(\vec{\theta})\xi = \xi^\dagger\sigma^0\xi, \tag{26}$$

while the spatial components do change (we use $\{\sigma^i,\sigma^j\} = 2\delta^{ij}$ and $\sigma^i\sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$)

$$\begin{aligned}
\xi^\dagger\sigma^i\xi &\to \xi^\dagger\left[\cos\frac{\theta}{2} - i\frac{\vec{\sigma}\cdot\vec{\theta}}{\theta}\sin\frac{\theta}{2}\right]\sigma^i\left[\cos\frac{\theta}{2} + i\frac{\vec{\sigma}\cdot\vec{\theta}}{\theta}\sin\frac{\theta}{2}\right]\xi \\
&= \xi^\dagger\left[\cos\frac{\theta}{2} - i\frac{\vec{\sigma}\cdot\vec{\theta}}{\theta}\sin\frac{\theta}{2}\right]\left(\left[\cos\frac{\theta}{2} - i\frac{\vec{\sigma}\cdot\vec{\theta}}{\theta}\sin\frac{\theta}{2}\right]\sigma^i + 2i\frac{\theta^i}{\theta}\sin\frac{\theta}{2}\right)\xi \\
&= \xi^\dagger\left(\left[\cos\theta - i\frac{\vec{\sigma}\cdot\vec{\theta}}{\theta}\sin\theta\right]\sigma^i + 2i\left[\cos\frac{\theta}{2} - i\frac{\vec{\sigma}\cdot\vec{\theta}}{\theta}\sin\frac{\theta}{2}\right]\frac{\theta^i}{\theta}\sin\frac{\theta}{2}\right)\xi \\
&= \xi^\dagger\left(\sigma^i\cos\theta - i(\delta^{ij} - i\epsilon^{ijk}\sigma^k)\frac{\theta^j}{\theta}\sin\theta + i\frac{\theta^i}{\theta}\sin\theta + 2\frac{\vec{\sigma}\cdot\vec{\theta}}{\theta}\frac{\theta^i}{\theta}\sin^2\frac{\theta}{2}\right)\xi
\end{aligned}$$

$$= \xi^\dagger \left(\sigma^i \cos\theta - \epsilon^{ijk}\sigma^k \frac{\theta^j}{\theta} \sin\theta + \frac{\vec{\sigma}\cdot\vec{\theta}}{\theta}\frac{\theta^i}{\theta}(1-\cos\theta) \right) \xi$$

$$= \xi^\dagger \left(\sigma^j \left[\frac{\theta^j\theta^i}{\theta^2} + \left(\delta^{ij} - \frac{\theta^j\theta^i}{\theta^2} \right) \cos\theta \right] - \epsilon^{ijk}\sigma^k \frac{\theta^j}{\theta}\sin\theta \right) \xi. \qquad (27)$$

Therefore, the component parallel to $\vec{\theta}$ remains unchanged, while the components orthogonal to $\vec{\theta}$ are rotated by the angle $\theta = |\vec{\theta}|$. If you want to see it more explicitly, you can specialize it to, *e.g.*, $\vec{\theta} = (0,0,\theta)$, and identify the usual rotation on the x-y plane. This way, we see that both time and spatial components transform properly under rotation.

Under a Lorentz boost, the time component transforms as

$$\begin{aligned} \xi^\dagger \sigma^0 \xi &\rightarrow \xi^\dagger \left[\cosh\frac{\eta}{2} + \frac{\vec{\sigma}\cdot\vec{\eta}}{\eta}\sinh\frac{\eta}{2} \right] \sigma^0 \left[\cosh\frac{\eta}{2} + \frac{\vec{\sigma}\cdot\vec{\eta}}{\eta}\sinh\frac{\eta}{2} \right] \xi \\ &= \xi^\dagger \left[\sigma^0 \cosh\eta + \frac{\vec{\sigma}\cdot\vec{\eta}}{\eta}\sinh\eta \right] \xi. \end{aligned} \qquad (28)$$

It leaves the time component boosted by $\cosh\eta$, together with the spatial component parallel to $\vec{\eta}$ by $\sinh\eta$. The spatial components transform as

$$\xi^\dagger \sigma^i \xi \rightarrow \xi^\dagger \left[\cosh\frac{\eta}{2} + \frac{\vec{\sigma}\cdot\vec{\eta}}{\eta}\sinh\frac{\eta}{2} \right] \sigma^i \left[\cosh\frac{\eta}{2} + \frac{\vec{\sigma}\cdot\vec{\eta}}{\eta}\sinh\frac{\eta}{2} \right] \xi$$

$$= \xi^\dagger \left[\cosh\frac{\eta}{2} + \frac{\vec{\sigma}\cdot\vec{\eta}}{\eta}\sinh\frac{\eta}{2} \right] \left(\left[\cosh\frac{\eta}{2} - \frac{\vec{\sigma}\cdot\vec{\eta}}{\eta}\sinh\frac{\eta}{2} \right] \sigma^i + 2\frac{\eta^i}{\eta}\sinh\frac{\eta}{2} \right) \xi$$

$$= \xi^\dagger \left(\sigma^i + \frac{\eta^i}{\eta}\sinh\eta + \sigma^j \frac{\eta^i\eta^j}{\eta^2}(\cosh\eta - 1) \right) \xi$$

$$= \xi^\dagger \left(\sigma^j \left[\left(\delta^{ij} - \frac{\eta^i\eta^j}{\eta^2} \right) + \frac{\eta^i\eta^j}{\eta^2}\cosh\eta \right] + \sigma^0 \frac{\eta^i}{\eta}\sinh\eta \right) \xi. \qquad (29)$$

It leaves the components orthogonal to $\vec{\eta}$ unchanged, while the parallel component is boosted by $\cosh\eta$ together with the time component by $\sinh\eta$. Combining time and spatial components, they therefore transform as a Lorentz four-vector as expected.

Similarly, out of χ in $(0, \frac{1}{2})$ representation, one can form a Lorentz four-vector by

$$\chi^\dagger \bar{\sigma}^\mu \chi, \qquad \bar{\sigma}^\mu = (1, -\vec{\sigma}). \qquad (30)$$

The spinors ξ and χ are right-handed and left-handed chirality spinors written in the two-component notation. This notation is used often in the study of supersymmetry.

Another possible candidate for spin one particle is to use $(1,0)$ and $(0,1)$ representations. We need both of them together because they are related by hermitean conjugation. I do not show it explicitly here, but you can verify that they correspond to field-strength tensors $F_{\mu\nu}$. There are six components for them as opposed to four components for a Lorentz vector, and the description is more redundant. Indeed, we do not regard $F_{\mu\nu}$ fundamental in gauge theories and rather use the four-vector A_μ as fundamental variables to describe spin one particles.

4 Gauge Symmetry

According to the discussion of the little group, massless spin one particle has $D-2$ degrees of freedom. The smallest field is a Lorentz vector A^μ with D components. The only way to reconcile this discrepancy is the gauge symmetry. It allows us to impose the Lorentz gauge condition $\partial_\mu A^\mu = 0$, which further leaves an additional invariance $A^\mu \to A^\mu + \partial^\mu \chi$ if $\partial^2 \chi = 0$. In the momentum space, $A^\mu = \epsilon^\mu(k) e^{-ik\cdot x}$ for $k^2 = 0$. Therefore, $k_\mu \epsilon^\mu(k) = 0$ and we can change $\epsilon^\mu(k) \to \epsilon^\mu(k) + k^\mu \chi$. Choosing $k^\mu = E(1, 1, 0, \cdots, 0)$, the Lorentz gauge condition requires $\epsilon^0 = \epsilon^1$. However the further gauge transformation allows us to eliminate both $\epsilon^0 = \epsilon^1$. Then the remaining non-vanishing components are $\epsilon^2, \cdots, \epsilon^{D-1}$, and hence there are only $D-2$ degrees of freedom.