129A HW # 4 (due Oct 3)

Use the Heisenberg equation of motion for operators O

$$i\hbar\frac{d}{dt}\mathcal{O} = i\hbar\frac{d}{dt}\mathcal{O}\Big|_{\text{explicit}} + [\mathcal{O}, H] \tag{1}$$

where the Hamiltonian is that for the Dirac particle

$$H = c(\vec{\alpha}\cdot\vec{\mathbf{p}} + mc\beta) \tag{2}$$

and the "explicit" part of the time-derivative is necessary if the operator $\mathcal{O}$ has an explicit dependence on t. Answer the following questions.

1. Show that the momentum $\vec{\mathbf{p}} = -i\hbar(\partial/\partial\vec{x})$ is conserved.

2. Show that the orbital angular momentum $\vec{L} = \vec{x}\times\vec{\mathbf{p}}$ is not conserved.

3. Show that the spin angular momentum

$$\vec{S} = \frac{\hbar}{2}\vec{\Sigma} = \frac{\hbar}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{3}$$

is not conserved, but the total angular momemtum $\vec{J} = \vec{L} + \vec{S}$ is conserved. (Notation changed from the class: Σ is used for 4×4 matrices, while σ always for Pauli matrices to avoid possible confusions.)

4. Show that the helicity $h = (\vec{s}\cdot\vec{\mathbf{p}})/|\vec{\mathbf{p}}|$ is conserved.

5. Show that the possible eigenvalues of the helicity are $\pm\hbar/2$.

6. Show that the above $\vec{\Sigma}$ matrix can be written in a Lorentz-covariant way,

$$\Sigma^i = \frac{i}{2}\epsilon^{ijk}[\gamma^j, \gamma^k]. \tag{4}$$

7. Guess the Lorentz-covariant generalization $M^{\mu\nu}$ of the angular momentum $M^{ij} = \epsilon^{ijk}J^k$. Show that M^{0i} are also conserved.