129B Solutions to HW#1

- **1. Decay rate has a dimension of $\frac{1}{[\text{Time}]}\left(\frac{(\#\text{ of events})}{\Delta t}\right)$.**

  In the natural units time has the dimension of inverse mass. (Think of the Uncertainty Principle $\Delta E\ \Delta t \sim \hbar$ and remember that $\hbar = 1$ $\longrightarrow [\text{Time}] = \frac{1}{[\text{Energy}]} = (\text{think of } E = mc^2,\ c = 1) = \frac{1}{[\text{Mass}]}$).

  Thus, decay rate must have the dimension of mass.

  Next,
  we know that decay rate is proportional to $|\text{Amplitude}|^2$,
  and the amplitude for the muon decay has G_F in it.
  Hence, $\Gamma \propto G_F^2$.

  Finally,
  to get the dimension right, we must find something with the dimension of $[\text{Mass}]^5$. We are neglecting the electron mass so the only mass parameter left is the muon mass M_μ, and so
  $\Gamma = (Constant) \times G_F^2\ M_\mu^5$.

- **2. The Particle Data Group booklet lists the following values for the lifetimes and branching fractions:**

  $\tau(\mu) = (2.19703 \pm 0.00004) \times 10^{-6}$ sec
  $\tau(\tau) = (291.0 \pm 1.5) \times 10^{-15}$ sec
  Branching Fraction $(\tau \longrightarrow e\,\bar{\nu}_e\,\nu_\tau) = (17.83 \pm 0.08)\ \%$
  Branching Fraction $(\tau \longrightarrow \mu\,\bar{\nu}_\mu\,\nu_\tau) = (17.35 \pm 0.10)\ \%$

  Notice that the relative error for the muon decay is much smaller. Indeed, it must be a lot easier to measure something with the lifetime of 2.2 microseconds, as opposed to 2.9×10^{-13} seconds! (Some of you may have even done the muon lifetime measurement in your 111 Lab class.) That is why the problem asks you to include the error bars only on the $\tau-$decays.

  The corresponding decay rates are:

  $$\Gamma^{\mu e} = \frac{1}{2.19703 \times 10^{-6}\ \text{sec}} = 455160\ \text{sec}^{-1}$$
  $$\Gamma^{\tau e} = \frac{1}{2.91 \times 10^{-13}\ \text{sec}} \times .1783 = 6.127 \times 10^{11}\ \text{sec}^{-1}$$

$$\Gamma^{\tau\mu} = \frac{1}{2.91\times 10^{-13}\ \text{sec}} \times .1735 = 5.962\times 10^{11}\ \text{sec}^{-1}$$

The error bars for the latter two :

$$\sigma\,(\Gamma^{\tau e}) =$$

$$\Gamma^{\tau e}\sqrt{\left(\frac{\sigma\,(\text{Br. ratio})}{(\text{Br. ratio})}\right)^2 + \left(\frac{\sigma\,(\tau)}{\tau}\right)^2} = 6.127\times 10^{11}\sqrt{\left(\frac{.08}{17.83}\right)^2 + \left(\frac{1.5}{291}\right)^2} = 4.19\times 10^{9}\ \text{sec}^{-1}$$

$$\sigma\,(\Gamma^{\tau\mu}) =$$

$$\Gamma^{\tau\mu}\sqrt{\left(\frac{\sigma\,(\text{Br. ratio})}{(\text{Br. ratio})}\right)^2 + \left(\frac{\sigma\,(\tau)}{\tau}\right)^2} = 5.962\times 10^{11}\sqrt{\left(\frac{.10}{17.35}\right)^2 + \left(\frac{1.5}{291}\right)^2} = 4.61\times 10^{9}\ \text{sec}^{-1}$$

- **3. Finally, consider the ratios :**

$$\frac{G_F^{\tau e}}{G_F^{\mu e}} = \left(\frac{\Gamma^{\tau e}}{M_\tau^5}\ \frac{M_\mu^5}{\Gamma^{\mu e}}\right)^{1/2} = \left(\frac{6.127\times 10^{11}}{455160}\ \frac{105.66^5}{1777^5}\right)^{1/2} = 1.00023$$

$$\frac{G_F^{\tau\mu}}{G_F^{\mu e}} = \left(\frac{\Gamma^{\tau\mu}}{M_\tau^5}\ \frac{M_\mu^5}{\Gamma^{\mu e}}\right)^{1/2} = \left(\frac{5.962\times 10^{11}}{455160}\ \frac{105.66^5}{1777^5}\right)^{1/2} = 0.98667$$

The error bars for the ratios :

$$\sigma\left(\frac{G_F^{\tau e}}{G_F^{\mu e}}\right) =$$

$$\sqrt{\left(\frac{1}{2}\ \frac{\sigma\,(\Gamma^{\tau e})}{\Gamma^{\tau e}}\right)^2 + \left(5\ \frac{\sigma\,(M_\tau)}{M_\tau}\right)^5} = \sqrt{\left(\frac{1}{2}\ \frac{4.19\times 10^{9}}{6.127\times 10^{11}}\right)^2 + \left(5\ \frac{.30}{1777}\right)^5} = 0.0035$$

$$\sigma\left(\frac{G_F^{\tau\mu}}{G_F^{\mu e}}\right) = \sqrt{\left(\frac{1}{2}\ \frac{\sigma\,(\Gamma^{\tau\mu})}{\Gamma^{\tau\mu}}\right)^2 + \left(5\ \frac{\sigma\,(M_\tau)}{M_\tau}\right)^5} = \sqrt{\left(\frac{1}{2}\ \frac{4.61\times 10^{9}}{5.962\times 10^{11}}\right)^2 + \left(5\ \frac{.30}{1777}\right)^5} = 0.0040$$

(***Hint :*** **if** $y = f\,(x, y)$, **then**

$$(\sigma\,(y))^2 = \left(\frac{\partial f}{\partial x}\right)^2 (\sigma\,(x))^2 + \left(\frac{\partial f}{\partial y}\right)^2 (\sigma\,(y))^2 .)$$

As the above numbers show, the error is dominated by the uncertainty in Γ, the uncertainty due to the error bar on the mass of the tau is almost negligible.

Putting everything together, we have

$$\frac{G_F^{\tau e}}{G_F^{\mu e}} = 1.00023 \pm 0.0035 \ \text{and}\ \frac{G_F^{\tau\mu}}{G_F^{\mu e}} = 0.98667 \pm 0.0040$$

The first ratio is in perfect agreement with 1, the second is very close to 1, but not within the error bar. Is there a problem? The answer is very instuctive : what you are seeing is the effect of the finite muon mass. The problem says that the correction due to the finite mass of the electron is of the order of

$\frac{M_e^2}{M_\mu^2}$. While the ratios $\frac{M_e^2}{M_\mu^2}$ and $\frac{M_e^2}{M_\tau^2}$ are indeed negligible, the ratio $\frac{M_\mu^2}{M_\tau^2}$ gives the correction of the order of 1 %. (The ratio itself actually equals 0.35 %, but you are not given the exact coefficient in front of it. The cofficient, in fact, equals (-8), so that the resulting correction is $0.35\,\% \times (-8) = -2.8\,\%$.) You can also understand the sign of the correction: $\Gamma^{\tau\mu}$ is smaller because in this case less phase space is available.

Optional

It you would like to compare not only the ratios but also the absolute values of the decay rate with experiment, you need to know the coefficient in front of $G_F^2 M_\mu^5$, and for that you have to do an honest calculation.

$$\Gamma = \int d\Gamma =$$

$$\int \frac{1}{2 M_\mu} \times (\text{Average } |\mathcal{M}|^2) \times d\Phi = \frac{1}{2 M_\mu} \int \frac{1}{2} \left(\sum_{hel} |\mathcal{M}|^2 \right) \times d\Phi$$

Luckily, $\sum_{hel} |\mathcal{M}|^2$ is given in the problem, it is equal to $128\, G_F^2 (p_{\nu_\mu} \cdot p_e)(p_{\nu_e} \cdot P)$.

The task is to evaluate the phase space intergral:

$$\int \frac{1}{2} \left(\sum_{hel} |\mathcal{M}|^2 \right) d\Phi =$$

$$\int \frac{1}{2} \left(\sum_{hel} |\mathcal{M}|^2 \right) (2\pi)^4 \delta^{(4)}(P - p_{\nu_e} - p_e - p_{\nu_\mu}) \times$$

$$\frac{d^3 \vec{p}_{\nu_\mu}}{(2\pi)^3 (2 E_{\nu_\mu})} \frac{d^3 \vec{p}_{\nu_e}}{(2\pi)^3 (2 E_{\nu_e})} \frac{d^3 \vec{p}_e}{(2\pi)^3 (2 E_e)}$$

The physical meaning of this intergral is that you are summing over all phase space points that satisfy the energy − momentum conservation constraint. The procedure here is the following: first you have to do 4 integrations to get rid of all 4 $\delta-$ functions and then do the remaining integrals. In this case we are neglecting the masses of the final state particles, so the intergration over $\delta-$ functions is really easy, you just have to remember 2 things:

- after you intergrate over some variable its value everywhere in the intergralshould be replaced by whatever the $\delta-$ function requires,

  i.e. $\int f(x)\, \delta(x - a)\, dx = f(a)$; and,

- you should keep in mind the rule

$\delta(f(x)) = 1/\left|\, f'(x)_{\text{at point } x_0 \text{ where } f(x_0)=0}\right|$.

To save space, we will not write $\frac{1}{2}\left(\sum_{hel} |\mathcal{M}|^2\right)$ in the intergrand for a while.

First, integrate over $\vec{p}_{\nu_\mu}$:

$$\int d\Phi = \frac{1}{8(2\pi)^5}\int \delta(M_\mu - E_{\nu_e} - E_e - E_{\nu_\mu})\frac{1}{E_{\nu_\mu}}\frac{d^3\vec{p}_{\nu_e}}{E_{\nu_e}}\frac{d^3\vec{p}_e}{E_e}.$$

In this expression now by E_{ν_μ} we mean

$$\sqrt{M_\mu^2 + \vec{p}_{\nu_\mu}^{\,2}} = |\vec{p}_{\nu_\mu}| = |\vec{p}_{\nu_e} + \vec{p}_e|.\ (M_{\nu_\mu} = 0).$$

Next, go to polar coordinates:

$$d^3\vec{p}_{\nu_e}\cdot d^3\vec{p}_e \longrightarrow p_{\nu_e}^2\, dp_{\nu_e}\, d\Omega_1 \cdot p_e^2\, dp_e\, d\Omega_2 =$$
$$E_{\nu_e}^2\, dE_{\nu_e}\, d\Omega_1 \cdot E_e^2\, dE_e\, d\Omega_2.\ (\text{Again, neglect } m_e).$$

$$\int d\Phi = \frac{1}{8(2\pi)^5}\int \delta(M_\mu - E_{\nu_e} - E_e - E_{\nu_\mu})\frac{1}{E_{\nu_\mu}} E_{\nu_e}\, dE_{\nu_e}\, d\Omega_1\, E_e\, dE_e\, d\Omega_2$$

Let's deal with the solid angle integral:

$$\int d\Omega_1\, d\Omega_2 = \int d\Omega\, d\phi\, d(\cos\theta),$$

where Ω specifies the direction of the first vector, and ϕ and θ specify the direction of the second vector relative to the first. This makes sence because the integrand depends only on the relative angle θ between the two vectors. In fact, the dependence is only in

$$E_{\nu_\mu} = |\vec{p}_{\nu_e} + \vec{p}_e| = (p_{\nu_e}^2 + 2\,p_{\nu_e}\,p_e\cos\theta + p_e^2)^{1/2} = (E_{\nu_e}^2 + 2\,E_{\nu_e}\,E_e\cos\theta + E_e^2)^{1/2},$$

because $\frac{1}{2}\left(\sum_{hel} |\mathcal{M}|^2\right) =$

$$64\,G_F^2\,(p_{\nu_\mu}\cdot p_e)(p_{\nu_e}\cdot P) = 32\,G_F^2\,(p_{\nu_\mu} + p_e)^2\,(p_{\nu_e}\cdot P) = 32\,G_F^2\,(P - p_{\nu_e})^2\,(p_{\nu_e}\cdot P) =$$
$$32\,G_F^2\,(P^2 - 2\,P\,p_{\nu_e} + p_{\nu_e}^2)(p_{\nu_e}\cdot P) = 32\,G_F^2\,(M_\mu^2 - 2\,M_\mu\,E_{\nu_e})\,M_\mu\,E_{\nu_e}$$ depends only on E_{ν_e}.

Hence, integration over Ω and ϕ is trivial:

$$\int d\Omega\, d\phi = 4\pi\times 2\pi = 8\pi^2.$$

Integration over $\cos\theta$ is a bit more interesting:

$$\int \delta(M_\mu - E_{\nu_e} - E_e - E_{\nu_\mu})\frac{1}{E_{\nu_\mu}} E_{\nu_e}\, E_e\, d(\cos\theta) = \frac{1}{(|dE)_{\nu_\mu} / d(\cos\theta)|}\frac{E_{\nu_e}\, E_e}{E_{\nu_\mu}} =$$
$$\left(\frac{2\,E_{\nu_e}\,E_e}{2\,(E_{\nu_e}^2 + 2\,E_{\nu_e}\,E_e\cos\theta + E_e^2)^{1/2}}\right)^{-1}\frac{E_{\nu_e}\,E_e}{E_{\nu_\mu}} = \frac{E_{\nu_\mu}}{E_{\nu_e}\,E_e}\,\frac{E_{\nu_e}\,E_e}{E_{\nu_\mu}} = 1.$$

Thus, we ended up with

$$\frac{8\pi^2}{8(2\pi)^5}\int \frac{1}{2}\left(\sum_{hel} |\mathcal{M}|^2\right) dE_{\nu_e}\, dE_e =$$
$$\frac{1}{32\pi^3}\int \frac{1}{2}\left(\sum_{hel} |\mathcal{M}|^2\right) dE_{\nu_e}\, dE_e.$$

We have been sloppy about the limits of the integration, now is the time to fix that. Consider energy – momentum conservation :

$$E_{\nu_\mu} = M_\mu - E_e - E_{\nu_e}$$

and $$E_{\nu_\mu} = |\vec{p}_{\nu_e} + \vec{p}_e| = (E_{\nu_e}^2 + 2 E_{\nu_e} E_e \cos\theta + E_e^2)^{1/2}$$

$$(M_\mu - E_e - E_{\nu_e})^2 = (E_{\nu_e}^2 + 2 E_{\nu_e} E_e \cos\theta + E_e^2)$$

$$M_\mu^2 - 2 (E_e + E_{\nu_e}) M_\mu + 2 E_{\nu_e} E_e (1 - \cos\theta) = 0$$

$$2 E_e (-M_\mu + E_{\nu_e} (1 - \cos\theta)) = 2 E_{\nu_e} M_\mu - M_\mu^2$$

$$E_e = \frac{M_\mu}{2} \frac{M_\mu - 2 E_{\nu_e}}{M_\mu - E_{\nu_e} (1 - \cos\theta)}$$

This means that for a given E_{ν_e} we have

$$M_\mu / 2 - E_{\nu_e} < E_e < M_\mu / 2,$$

as $\cos\theta$ changes from -1 to 1.

Finally, we can write

$$\Gamma = \frac{1}{32 \pi^3} \frac{1}{2 M_\mu} \int_0^{M_\mu/2} \frac{1}{2} \left(\sum_{hel} |\mathcal{M}|^2 \right) dE_{\nu_e} \int_{M_\mu/2 - E_{\nu_e}}^{M_\mu/2} dE_e =$$

$$\frac{1}{32 \pi^3} \frac{1}{2 M_\mu} \int_0^{M_\mu/2} 32 G_F^2 (M_\mu^2 - 2 M_\mu E_{\nu_e}) M_\mu E_{\nu_e} dE_{\nu_e} \times E_{\nu_e} =$$

$$\frac{G_F^2}{2 \pi^3} \int_0^{M_\mu/2} (M_\mu^2 - 2 M_\mu E_{\nu_e}) E_{\nu_e}^2 dE_{\nu_e} =$$

$$\frac{G_F^2}{2 \pi^3} \left(M_\mu^2 \frac{(M_\mu / 2)^3}{3} - 2 M_\mu \frac{(M_\mu / 2)^4}{4} \right) = \frac{G_F^2 M_\mu^5}{2 \times 8 \times 12 \pi^3} = \frac{G_F^2 M_\mu^5}{192 \pi^3}$$ ☺